Discrete torsion

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In this article we explain discrete torsion. Put simply, discrete torsion is the choice of orbifold group action on the B field. We derive the classification $H^2(\Gamma, U(1))$, the twisted sector phases appearing in string loop partition functions, Douglas's description of discrete torsion for D-branes in terms of a projective representation of the orbifold group, and outline how the results of Vafa and Witten fit into this framework. In addition, we observe that additional degrees of freedom (known as shift orbifolds) appear in describing orbifold group actions on B fields, in addition to those classified by $H^2(\Gamma, U(1))$, and explain how these degrees of freedom appear in terms of twisted sector contributions to partition functions and in terms of orbifold actions on D-brane worldvolumes. This paper represents a technically simplified version of prior papers by the author on discrete torsion. We repeat here technically simplified versions of results from those papers, and have included some new material.

I. INTRODUCTION

Historically discrete torsion has been a rather mysterious aspect of string theory. Discrete torsion was originally discovered [1] as an ambiguity in the choice of phases to assign to twisted sectors of string orbifold partition functions. Although other work has been done on the subject (see, for example, [2–4]), no work done prior to [5,6] has succeeded in giving any sort of genuinely deep understanding of discrete torsion. In fact, discrete torsion as sometimes been referred to has an inherently stringy degree of freedom, without any geometric analogue.

In this paper (a followup to [5,6]) we shall describe a purely mathematical way of understanding discrete torsion, and will show explicitly how our description gives rise to Vafa's phases in twisted sector contributions to partition functions, to projective representations of orbifold group actions on D-branes, and to other physical manifestations of discrete torsion.

The description of discrete torsion we present here is the same as that we previously presented in [5,6]. This paper differs in that we have vastly reduced the level of technical complication that was present in [5,6], we explicitly work out the details of some computations merely referred to in [5,6], and also work out some new results not present in [5,6], such as a derivation of the projectivized group actions used by [3,4] to describe orbifold group actions on D-branes with discrete torsion.

What is discrete torsion? In a nutshell, *discrete torsion is the choice of orbifold group action on the B field.*

More generally, in any theory possessing fields with gauge invariances, defining the orbifold group action on the base space does not suffice to define the orbifold group action on the fields of the theory—one can combine the action of the orbifold group with gauge transformations to get new, distinct, actions of the orbifold group. For $U(1)$ gauge fields, this gives rise to orbifold $U(1)$ Wilson lines. For B fields,

this gives rise to discrete torsion, as we shall work through in detail in this paper. Now, string theory has other fields with gauge invariances—for example, in eleven-dimensional supergravity there is a three-form potential with a gauge invariance analogous to that of the B field, and so one expects to have orbifold degrees of freedom associated to that field also. We shall discuss analogues of discrete torsion for the other tensor-field potentials of string theory in [7]. In another upcoming paper [8] we shall discuss discrete torsion in perturbative heterotic strings.

It should be emphasized that we have presented a first-principles *explanation* of discrete torsion in [5,6] and in this paper—not some observations on discrete torsion, not some calculations related to discrete torsion, but an explanation. It should also be emphasized that this explanation is purely mathematical in nature—although we can certainly check physical consequences, at its heart discrete torsion is a natural mathematical consequence of having B fields. Discrete torsion has nothing at all to do with string theory *per se*, and is not "inherently stringy" in any sense.

We begin in Sec. II with a discussion of orbifold $U(1)$ Wilson lines—i.e., a discussion of counting orbifold group actions on $U(1)$ gauge fields. Although the technical details for orbifold group actions on B fields are considerably more complicated and subtle, the basic principles are the same for B fields as for $U(1)$ gauge fields, and our treatment of B fields parallels our treatment of $U(1)$ gauge fields. In particular, Sec. II provides a simplified context in which to see the main ideas at work.

Next in Sec. III we study orbifold group actions on B fields, and explain how the group cohomology group $H^2(\Gamma, U(1))$ arises. In addition to $H^2(\Gamma, U(1))$, we also find additional orbifold group actions; we explain why, in hindsight, such additional actions should be expected. The general methods used are the same as for studying orbifold group actions on $U(1)$ gauge fields, although the technical details are more complicated.

In Sec. IV we derive the twisted sector phases originally described in [1]. We calculate the phases at one-loop, and also check factorization at higher loops (an explicit calcula-

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tion is done at two-loops, from which the general result is obvious).

In Sec. V we derive Douglas's description [3,4] of discrete torsion for D-branes, as a projective representation of the orbifold group.

In Sec. VI we outline how results of Vafa and Witten [2] on the interplay of discrete torsion and Calabi-Yau moduli are naturally understood in the context we have presented. This discussion frequently refers to reflexive sheaves on singular varieties, and as such ideas are not frequently used in the physics literature, we have included an appendix on the subject.

Finally in Sec. VII we very briefly mention local orbifold degrees of freedom, as distinct from global orbifold degrees of freedom, a topic usually neglected in physics treatments of orbifolds.

Readers who wish to study this paper in detail are encouraged to first work through Sec. II on orbifold $U(1)$ Wilson lines, despite the fact that it might not sound wholly relevant. Our approach to thinking about B fields and discrete torsion is very closely related to orbifold $U(1)$ Wilson lines, and the mathematical techniques we shall use to study orbifold group actions on B fields are precise analogues of those used to classify orbifold $U(1)$ Wilson lines. Mathematicians should note that although these mathematical techniques are a staple of the relevant part of the mathematics literature, they are not widely used in the physics literature.

II. REVIEW OF ORBIFOLD $U(1)$ WILSON LINES

In this paper we shall describe discrete torsion as a choice of orbifold group action on B fields. In order to understand how this works, however, we shall first present orbifold $U(1)$ Wilson lines. We will work through this simpler case in detail because it is an exact model for our approach to understanding orbifold group actions on B fields—the details are much more subtle, but the basic ideas are the same as for orbifold $U(1)$ Wilson lines.

In particular, our approach to understanding discrete torsion is closely modelled on understanding orbifold $U(1)$ Wilson lines mathematically, involving techniques not commonly used in the physics literature. In order to understand later sections of this paper, therefore, it is important to first get a solid handle on the basic ideas, in a context in which the details are easy to work out.

Orbifold $U(1)$ Wilson lines are precisely a choice of orbifold group action on $U(1)$ gauge fields. We shall first work out a description of elements of the set of orbifold group actions on $U(1)$ gauge fields, then we shall argue that any two elements of the set differ by an element of the group $H^1(\Gamma, U(1))$.

Our analysis does not rely on the orbifold group Γ acting freely—whether Γ has fixed points is entirely irrelevant for our analysis. Similarly, whether Γ is Abelian is equally irrelevant.

It should also be mentioned that our analysis of orbifold actions on $U(1)$ gauge fields is not new, but is rather quite standard in the mathematics literature. The earliest reference of which we are aware is [9], Sec 1.13.

For simplicity, we shall work at the level of transition functions. A line bundle with connection [i.e., a set of local $U(1)$ gauge fields] can be described by pairs $(A^\alpha, g_{\alpha\beta})$ where A^α is a vector field on an element U_α of an open cover, and $g_{\alpha\beta}$ are transition functions. These are related by

$$A^\alpha - A^\beta = d \log g_{\alpha\beta}. \tag{1}$$

A. Orbifold group actions on vector fields

In this section we shall study the set of orbifold group actions on vector fields, in terms of transition functions and data defined on local coordinate patches. To describe the action of the orbifold group on such data, one relates pullbacks $g^* g_{\alpha\beta}$, for example, to the original transition functions $g_{\alpha\beta}$. So, in this section we shall work out relationships between pullbacks of transition functions $g^* g_{\alpha\beta}$ and gauge fields $g^* A^\alpha$, and original transition functions $g_{\alpha\beta}$ and gauge fields A^α.

For convenience, we shall choose an open cover that is well behaved with respect to the action of the orbifold group. Specifically, let $\{U_\alpha\}$ be a "good invariant" cover, meaning that each $g \in \Gamma$ maps each U_α back into itself (i.e., $g: U_\alpha \rightarrow U_\alpha$), and each U_α is a union of disjoint contractible open sets. Such a cover is not a good cover, because the elements U_α will not be contractible in general, but is the next best thing, and suffices for our purposes.

To begin, we need to demand that the bundle is isomorphic to itself under pullback by group elements, i.e., the bundle itself is "symmetric" with respect to the group action. Given this constraint, we will derive the form of the pullback of the $U(1)$ gauge field from self-consistency. At the level of transition functions, this is the statement

$$g^* g_{\alpha\beta} = (h_\alpha^g)(g_{\alpha\beta})(h_\beta^g)^{-1} \tag{2}$$

for some Čech cochains h_α^g. (Such cochains define an isomorphism from the bundle itself to its pullback by g.) If such a statement were not true, one could not even begin to define an orbifold group action, as it would mean that in no sense is the bundle well behaved with respect to the orbifold group.

Next, we need to determine how $h_\alpha^{g_1 g_2}$ is related to $h_\alpha^{g_1}$ and $h_\alpha^{g_2}$. We can find such a constraint by expanding $(g_1 g_2)^* g_{\alpha\beta}$ in two different ways:

$$(g_1 g_2)^* g_{\alpha\beta} = \begin{cases} (h_\alpha^{g_1 g_2})(g_{\alpha\beta})(h_\beta^{g_1 g_2})^{-1} \\ g_2^* [(h_\alpha^{g_1})(g_{\alpha\beta})(h_\beta^{g_1})^{-1}] \\ = (g_2^* h_\alpha^{g_1})(h_\alpha^{g_2})(g_{\alpha\beta})(h_\beta^{g_2})^{-1}(g_2^* h_\beta^{g_1})^{-1}. \end{cases}$$

From self-consistency, we see that it is natural to demand

$$h_\alpha^{g_1 g_2} = (g_2^* h_\alpha^{g_1})(h_\alpha^{g_2}). \tag{3}$$

We should take a moment to comment on this "derivation." All we really know is that the Čech coboundary of the h_α^g satisfies an equation of the form above—strictly speaking, it is not quite true that Eq. (3) necessarily follows. However, we are looking for constraints of the general form of Eq. (3),

and from the previous algebra Eq. (3) emerges quite naturally. If the reader prefers, it might be slightly more fair to say that we are using self-consistency to bootstrap an ansatz. We shall use similar methods many more times, both here and in working out orbifold group actions on B fields.

In addition, in this special case, there is an additional concern that the reader might have. In principle, we could multiply one side of Eq. (3) by an α-independent phase, and the result would still be consistent. We are implicitly imposing a slightly stronger constraint than strictly necessary—namely, that the orbifold group action be honestly represented. This choice is precisely the choice that leads to the calculation of orbifold Wilson lines, for example.

In passing, note that Eq. (3) can be viewed as the statement that the map from the orbifold group Γ to bundle isomorphisms (as defined by Čech cochains h_α^g) is a group homomorphism.

At this point we have derived the form of an equivariant structure on the principal $U(1)$ bundle itself, but have not mentioned the connection on the bundle. Without loss of generality, define φ_g^α by

$$g^*A^\alpha = A^\alpha + \varphi_g^\alpha.$$

Certainly, regardless of g^*A^α, we can write this for some φ_g^α, so all we have done is define φ_g^α, not place any sort of constraint on the connection A^α. By expanding $(g_1 g_2)^*A^\alpha$ in two different ways, one quickly finds

$$\varphi_{g_1 g_2}^\alpha = \varphi_{g_2}^\alpha + g_2^*\varphi_{g_1}^\alpha.$$

By pulling back Eq. (1) by g, one finds

$$\varphi_g^\alpha = d\log h_\alpha^g. \tag{4}$$

So far we have worked out how to describe the action of an orbifold group Γ on a principal $U(1)$ bundle with connection [a set of $U(1)$ gauge fields, if the reader prefers]. To summarize our results so far, we can describe this action at the level of transition functions as

$$g^*A^\alpha = A^\alpha + d\log h_\alpha^g$$

$$g^*g_{\alpha\beta} = (h_\alpha^g)(g_{\alpha\beta})(h_\beta^g)^{-1},$$

$$h_\alpha^{g_1 g_2} = (h_\alpha^{g_2})(g_2^*h_\alpha^{g_1})$$

for some Čech cochains h_α^g, which define the orbifold group action.

We have described the action of the orbifold group in terms of transition functions and data on local charts, but one can also describe the same orbifold group action more elegantly in terms of an action on the total space of a bundle. See for example [5] where this approach is reviewed.

B. Differences between orbifold group actions

In the previous section we described elements of the set of orbifold group actions on $U(1)$ gauge fields [more properly, principal $U(1)$ bundles with connection]. We have not used

the term set loosely—in general, there is no natural way to place a group structure, for example, on this set. In this section we shall point out that any two such actions (i.e., any two elements of the set) differ by a constant gauge transformation, and those constant gauge transformations lead to $H^1(\Gamma, U(1))$.

Let h_α^g be one set of Čech cochains describing an orbifold action on a set of $U(1)$ gauge fields, and let $\bar{h}_\alpha^g$ be a distinct orbifold action on the same set of $U(1)$ gauge fields.

Define Čech cochains ϕ_α^g by

$$\phi_\alpha^g = \frac{h_\alpha^g}{\bar{h}_\alpha^g}.$$

From writing $g^*g_{\alpha\beta}$ in terms of the two orbifold actions, we find a constraint on the ϕ_α^g:

$$g^*g_{\alpha\beta} = \begin{cases} (h_\alpha^g)(g_{\alpha\beta})(h_\beta^g)^{-1} \\ (\bar{h}_\alpha^g)(g_{\alpha\beta})(\bar{h}_\beta^g)^{-1}. \end{cases}$$

Dividing these two lines we find that $\phi_\alpha^g = \phi_\beta^g$ on $U_\alpha \cap U_\beta$, i.e., the ϕ_α^g define a function, which we shall denote ϕ^g.

Next, write g^*A^α in terms of the two orbifold group actions to find an additional constraint:

$$g^*A^\alpha = \begin{cases} A^\alpha + d\log h_\alpha^g \\ A^\alpha + d\log \bar{h}_\alpha^g. \end{cases}$$

Subtracting these two lines, we find that

$$d\log \phi^g = 0.$$

In other words, ϕ^g is a constant function.

Finally, from

$$h_\alpha^{g_1 g_2} = (h_\alpha^{g_2})(g_2^*h_\alpha^{g_1})$$

$$\bar{h}_\alpha^{g_1 g_2} = (\bar{h}_\alpha^{g_2})(g_2^*\bar{h}_\alpha^{g_1})$$

we find that

$$\phi^{g_1 g_2} = \phi^{g_2} g_2^*\phi^{g_1}. \tag{5}$$

Assuming the covering space of the orbifold is connected, we see that the ϕ define a group homomorphism $\Gamma \to U(1)$.

In other words, the difference between any two orbifold group actions on $U(1)$ gauge fields, on a connected space, is defined by an element of $H^1(\Gamma, U(1))$.

C. General analysis

In the previous two sections we did two things—we worked out the structure of an orbifold group action on a set of $U(1)$ gauge fields, and then we argued that any two orbifold group actions differ by a (constant) gauge transformation, defining an element of $H^1(\Gamma, U(1))$.

One point mentioned earlier, and worth emphasizing, is that our derivation of the group $H^1(\Gamma, U(1))$ did not rely

upon Γ being freely acting—the derivation is the same regardless of whether or not the action of the orbifold group Γ has fixed points. The same statement will be true of our derivation of $H^2(\Gamma, U(1))$ in understanding orbifold group actions on B fields in the next section—our derivation holds regardless of whether or not Γ acts freely.

Another point worth mentioning is that we have not assumed that Γ is Abelian—our derivation of $H^1(\Gamma, U(1))$ is the same regardless of whether Γ is Abelian or non-Abelian.

We should also note that we have not assumed that the principal $U(1)$ bundle on which we have defined the orbifold group action is trivial. If the bundle is nontrivial, then one does not expect to always be able to define an orbifold group action, even on the topological bundle, much less on the bundle with connection. For example, consider the Hopf fibration of S^3 over S^2, viewed as a principal $U(1)$ bundle over S^2. Consider a $U(1)$ acting on the S^2 by rotations about some fixed axis. A 2π rotation of the S^2 is the same as the identity action on S^2; however, a 2π rotation of the S^2 does not lift to the action of the identity on S^3—one must rotate the S^2 by a multiple of 4π instead, as is discussed in most elementary quantum mechanics textbooks in the context of spin. See for example [5] for further explanation of the standard well-known fact that group actions on base spaces do not always lift to nontrivial bundles. Assuming that group actions on the bundle exist, the difference between any two group actions on a principal $U(1)$ bundle with connection is defined by an element of $H^1(\Gamma, U(1))$.

D. The set of orbifold group actions is a *set*

It should be emphasized that the set of orbifold group actions on a principal $U(1)$ bundle with connection is a *set*, and in general does not naturally have a group structure. Often in the physics literature, calculations boil down to calculating some cohomology group—by contrast, possible orbifold group actions do not (in general) have a group structure, and certainly cannot be understood in terms of a calculation of a cohomology group.

Now, in special cases, it is possible to put a group structure on the set of orbifold group actions. For example, if the principal $U(1)$ bundle is topologically trivial, then there is a natural notion of a trivial action—since the base and the fiber can be globally split, one could take the orbifold group to act on the base only. In this special case, we can describe any other orbifold group action in terms of the trivial action plus an element of $H^1(\Gamma, U(1))$—this is the precise technical meaning of "combining the action of the orbifold group with a gauge transformation," as is often mentioned in the old string orbifold literature.

In general, however, there will be no natural notion of a "trivial" action—all the orbifold group actions will have some nontrivial action on the fibers of the bundle, and so the set of orbifold group actions is no more than a set.[1] To be blunt, anyone who tries to understand orbifold group actions

on any gauge field in terms of any (cohomology) *group* at almost any level of generality will not get far.

A simple example of these notions is provided by line bundles on toric varieties. Specifying a specific toric divisor is the same as specifying an action of the algebraic torus underlying the toric variety on the line bundle (see [10] and references therein). For example, consider $\mathbf{P}^2$ as a toric variety, with toric divisors generated by D_x, D_y, and D_z. Possible toric divisors for a degree 0 bundle on $\mathbf{P}^2$ include $D_x - D_y$, $2D_z - D_x - D_y$, and so forth—countably many, counted by[2] $H^1((\mathbf{C}^\times)^n, \mathbf{C}^\times) = \mathbf{Z}^n$. For a degree 0 line bundle, one can put a group structure on the set of degree 0 divisors—take the identity to be the toric divisor 0. Now, consider a degree 1 line bundle. Possible toric divisors include D_x, D_y, $3D_x - 2D_z$, and so forth—countably many, counted by $H^1((\mathbf{C}^\times)^n, \mathbf{C}^\times) = \mathbf{Z}^n$. However, here there is no natural divisor to associate with the identity—the set does not have a group structure in any natural way. So, we see here explicitly that in general the set of orbifold group actions is only a set.

In special cases, such as when the bundle is topologically trivial, there is a canonical trivial orbifold group action, and in such cases we can put a group structure on the set of orbifold group actions, which becomes the group $H^1(\Gamma, U(1))$. As luck would have it, such special cases are the only ones ever usually considered by physicists, so most of the subtleties of the general case are omitted from typical physics discussions. These matters are discussed in more detail in [5].

In the next section we shall perform a closely analogous computation for B fields. We shall first study elements of the set of orbifold group actions on the B fields, then we shall study how different elements of this set are related. The technical details for B fields are much more complicated, but the basic approach is the same.

III. DERIVATION OF $H^2(\Gamma, U(1))$

In this section we shall explain how the group cohomology group $H^2(\Gamma, U(1))$ appears when describing orbifold group actions on B fields. Our methods will closely mirror standard methods used to study orbifold $U(1)$ Wilson lines, so readers are encouraged to study Sec. II before reading this section. To be brief, we first derive the structure of elements of the set of orbifold group actions on B fields, and then study the difference between any two elements of this set. The difference is a gauge transformation of B fields, just as the difference between two orbifold group actions on a set of $U(1)$ gauge fields is a gauge transformation.

A gauge transformation of a B field is defined by a principle $U(1)$ bundle with connection—a set of $U(1)$ gauge fields, if the reader prefers. So, to each element g of the orbifold group Γ, the difference between any two orbifold

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[1] A set naturally acted on by the group $H^1(\Gamma, U(1))$, but a set nonetheless.

[2] For holomorphic line bundles, rather than principal $U(1)$ bundles with connection, there is a closely analogous argument relating different choices of actions of algebraic groups. The result is essentially the same, modulo replacing $U(1)$ with $\mathbf{C}^\times$.

group actions on a set of B fields is defined by a principal $U(1)$ bundle with connection, call it T^g. [In fact, the connection on T^g is constrained to be flat, just as the difference between any two orbifold group actions on $U(1)$ gauge fields was a constant gauge transformation.] We must also specify isomorphisms $\omega^{g,h}$ between T^{gh} and [3] $T^h \otimes h^* T^g$ for each $g, h \in \Gamma$, just as for orbifold Wilson lines, the gauge transformations ϕ^g were constrained to obey $\phi^{g_1 g_2} = \phi^{g_2} \cdot g_2^* \phi^{g_1}$.

This description is not complete—there are "residual gauge invariances," specifically, only the isomorphism class of T^g is actually relevant. To get a precise counting, one must fix this residual gauge invariance.

Elements of $H^2(\Gamma, U(1))$ arise from taking the bundle T^g to be topologically trivial, with a connection that is gauge equivalent to zero. Using residual gauge invariances, we can replace T^g with the canonical trivial bundle with identically zero connection, and the maps $\omega^{g,h}$ become constant gauge transformations defining elements of $H^2(\Gamma, U(1))$.

A. Orbifold group actions on B fields

In this section we shall work out a description of elements of the set of orbifold group actions on B fields. As for $U(1)$ gauge fields, orbifold group actions on B fields can be determined by specifying how pullbacks are related to original data, so we shall be studying pullbacks.

It is important to emphasize at the start that we really do mean to use the word set—the set of orbifold group actions, in general, cannot canonically be given any group structure. Most mathematically oriented physics papers calculate cohomology groups or generalized cohomology groups—here, by contrast, we shall begin by calculating a set, which cannot be understood as a group in general, much less any sort of (generalized) cohomology group. After we have worked out this set, in the next section we shall argue that elements of this set differ by gauge transformations of B fields, which will lead us to discover $H^2(\Gamma, U(1))$.

Let $\{U_\alpha\}$ be a "good invariant" cover, as before. Then, a two-form field potential is described as a collection of two-forms B^α, one for each open set U_α, related by gauge transformation on overlaps.

We assume that the B field has no magnetic sources (meaning, that the exterior derivative of its curvature H vanishes), and that the curvature H is (the image of) an element of integral cohomology $H^3(\mathbf{Z})$. Such B fields on a smooth space X are described on the open cover $\{U_\alpha\}$ by [11,12] two-forms B^α on U_α, one-forms $A^{\alpha\beta}$ on $U_\alpha \cap U_\beta = U_{\alpha\beta}$, and $U(1)$-valued functions $h_{\alpha\beta\gamma}$ on $U_{\alpha\cap\beta\cap\gamma} = U_{\alpha\beta\gamma}$, satisfying

$$B^\alpha - B^\beta = dA^{\alpha\beta}$$

$$A^{\alpha\beta} + A^{\beta\gamma} + A^{\gamma\alpha} = d\log h_{\alpha\beta\gamma}$$

$$\delta(h_{\alpha\beta\gamma}) = 1.$$

We should mention that in writing the above we are not putting any more structure on B fields than is already present in string theory. For example, the reader might be concerned at the appearance of vector fields $A^{\alpha\beta}$; such a reader should be reminded that if we define B fields in open patches, then on overlaps the B fields will differ by some gauge transformation. By specifying the $A^{\alpha\beta}$ we have merely made the gauge transformations on overlaps explicit, no more.

In this section we shall work out how to describe orbifold group actions on B fields, in terms of the data above. To begin, demand that the Čech cocycles $h_{\alpha\beta\gamma}$ are preserved by the orbifold group, up to coboundaries. Specifically, demand

$$g^* h_{\alpha\beta\gamma} = h_{\alpha\beta\gamma} \nu^g_{\alpha\beta} \nu^g_{\beta\gamma} \nu^g_{\gamma\alpha} \tag{6}$$

for some Čech cochains ν^g, for each $g \in \Gamma$.

Next, we shall derive a constraint on the coboundaries ν^g from self-consistency of Eq. (6). Specifically,

$$(g_1 g_2)^* h_{\alpha\beta\gamma}$$

$$= \begin{cases} h_{\alpha\beta\gamma} \nu^{g_1 g_2}_{\alpha\beta} \nu^{g_1 g_2}_{\beta\gamma} \nu^{g_1 g_2}_{\gamma\alpha} \\ g_2^* (h_{\alpha\beta\gamma} \nu^{g_1}_{\alpha\beta} \nu^{g_1}_{\beta\gamma} \nu^{g_1}_{\gamma\alpha}) \\ = h_{\alpha\beta\gamma} (\nu^{g_2}_{\alpha\beta} g_2^* \nu^{g_1}_{\alpha\beta}) (\nu^{g_2}_{\beta\gamma} g_2^* \nu^{g_1}_{\beta\gamma}) (\nu^{g_2}_{\gamma\alpha} g_2^* \nu^{g_1}_{\gamma\alpha}) \end{cases}$$

from which we see that

$$\nu^{g_1 g_2}_{\alpha\beta} = \nu^{g_2}_{\alpha\beta} g_2^* \nu^{g_1}_{\alpha\beta} (h^{g_1,g_2}_\alpha)(h^{g_1,g_2}_\beta)^{-1} \tag{7}$$

for some Čech cochains h^{g_1,g_2}.

By applying Eq. (7) to expand $\nu^{g_1 g_2 g_3}$ in two different ways, we can derive

$$(h^{g_1,g_2 g_3}_\alpha)(h^{g_2,g_3}_\alpha) = (g_3^* h^{g_1,g_2}_\alpha)(h^{g_1 g_2,g_3}_\alpha). \tag{8}$$

Next, consider the forms B^α and $A^{\alpha\beta}$. Define two-forms $B(g)^\alpha$ and one-forms $A(g)^\alpha$ by

$$g^* B^\alpha = B^\alpha + B(g)^\alpha$$

$$g^* A^{\alpha\beta} = A^{\alpha\beta} + A(g)^{\alpha\beta}.$$

We shall use self-consistency to work out meaningful expressions for $B(g)^\alpha$ and $A(g)^{\alpha\beta}$.

From the expression

$$g^* (B^\alpha - B^\beta) = g^* dA^{\alpha\beta}$$

one can derive

$$B(g)^\alpha - B(g)^\beta = dA(g)^{\alpha\beta}. \tag{9}$$

Furthermore, by expanding both sides of the expression

$$g^* (A^{\alpha\beta} + A^{\beta\gamma} + A^{\gamma\alpha}) = g^* d\log h_{\alpha\beta\gamma}$$

we find

$$A(g)^{\alpha\beta} = d\log \nu^g_{\alpha\beta} + \Lambda(g)^\alpha - \Lambda(g)^\beta \tag{10}$$

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[3] We will explain the meaning of $\otimes$ for principal $U(1)$ bundles later in this section.

for some one-forms $\Lambda(g)^\alpha$ defined on open sets U_α. Comparing Eqs. (9) and (10), we find

$$B(g)^\alpha = d\Lambda(g)^\alpha. \tag{11}$$

By expanding $(g_1 g_2)^* B^\alpha$ in two different ways, we find

$$\Lambda(g_1 g_2)^\alpha = \Lambda(g_2)^\alpha + g_2^* \Lambda(g_1)^\alpha + d\Lambda^{(2)}(g_1, g_2)^\alpha \tag{12}$$

for some real-valued functions $\Lambda^{(2)}(g_1, g_2)^\alpha$ defined on the open sets U_α.

Finally, by expanding $(g_1 g_2)^* A^{\alpha\beta}$ in two different ways, we find that

$$d\Lambda^{(2)}(g_1, g_2)^\alpha = -d\log h_\alpha^{g_1, g_2}.$$

We can summarize the results of these computations as follows:

$$g^* B^\alpha = B^\alpha + d\Lambda(g)^\alpha$$

$$g^* A^{\alpha\beta} = A^{\alpha\beta} + d\log \nu_{\alpha\beta}^g + \Lambda(g)^\alpha - \Lambda(g)^\beta$$

$$\Lambda(g_1 g_2)^\alpha = \Lambda(g_2)^\alpha + g_2^* \Lambda(g_1)^\alpha - d\log h_\alpha^{g_1, g_2}$$

$$g^* h_{\alpha\beta\gamma} = h_{\alpha\beta\gamma} \nu_{\alpha\beta}^g \nu_{\beta\gamma}^g \nu_{\gamma\alpha}^g$$

$$\nu_{\alpha\beta}^{g_1 g_2} = (\nu_{\alpha\beta}^{g_2})(g_2^* \nu_{\alpha\beta}^{g_1})(h_\alpha^{g_1, g_2})(h_\beta^{g_1, g_2})^{-1}$$

$$(h_\alpha^{g_1, g_2 g_3})(h_\alpha^{g_2, g_3}) = (g_3^* h_\alpha^{g_1, g_2})(h_\alpha^{g_1 g_2, g_3})$$

where $\Lambda(g)^\alpha$, $\nu_{\alpha\beta}^g$, and $h_\alpha^{g_1, g_2}$ are structures introduced to define the action of the orbifold group on the B field.

We have defined orbifold group actions on B fields in terms of the transition functions and other local data defining the B field. More formally, a B field can also be understood as a connection on a "1-gerbe," a special kind of stack, or (loosely) sheaf of categories. We discussed such objects in [6], together with a discussion of how one defines orbifold group actions on them. Our discussion in [6] is adapted from [14], which discusses B fields in the language of stacks.

B. Differences between orbifold group actions

In describing orbifold $U(1)$ Wilson lines, the group $H^1(\Gamma, U(1))$ arises as differences between orbifold group actions; similarly, in describing discrete torsion, $H^2(\Gamma, U(1))$ arises in describing the differences between orbifold group actions. In both cases, one can get any action from any other action by combining the action with a set of gauge transformations; in the former case, $H^1(\Gamma, U(1))$ counts those gauge transformations, and in the latter case, $H^2(\Gamma, U(1))$ counts some of the gauge transformations.

One unusual matter we shall discover in this section is that, in addition to elements of $H^2(\Gamma, U(1))$, one sometimes has additional actions of the orbifold group on the B fields—sometimes, there may be more to "discrete torsion" than just $H^2(\Gamma, U(1))$. We shall show in a later section that, at the level of twisted sector contributions to partition functions, these additional contributions can do more than merely mul-

tiply contributions $Z_{(g,h)}$ by a phase; they significantly alter the $Z_{(g,h)}$ themselves. We shall discuss these elements in much greater detail later in this section.

Consider two distinct orbifold group actions on the B fields. Denote one orbifold group action as in the last section, and denote the second action with a bar, e.g., $\bar{\nu}_{\alpha\beta}^g$ rather than $\nu_{\alpha\beta}^g$.

Define

$$T_{\alpha\beta}^g = \frac{\nu_{\alpha\beta}^g}{\bar{\nu}_{\alpha\beta}^g}. \tag{13}$$

From dividing the expressions

$$g^* h_{\alpha\beta\gamma} = \begin{cases} (h_{\alpha\beta\gamma})(\nu_{\alpha\beta}^g)(\nu_{\beta\gamma}^g)(\nu_{\gamma\alpha}^g) \\ (h_{\alpha\beta\gamma})(\bar{\nu}_{\alpha\beta}^g)(\bar{\nu}_{\beta\gamma}^g)(\bar{\nu}_{\gamma\alpha}^g) \end{cases}$$

we see that

$$T_{\alpha\beta}^g T_{\beta\gamma}^g T_{\gamma\alpha}^g = 1$$

meaning that the T^g are transition functions for a principal $U(1)$ bundle.

The fact that we are seeing principal $U(1)$ bundles at the same place where gauge transformations appeared in describing orbifold $U(1)$ Wilson lines is no accident—a gauge transformation of a set of B fields is defined by a(n equivalence class of) principal $U(1)$ bundles with connection [6]. So, just as for orbifold Wilson lines, we are already seeing that the difference between two lifts is defined by a gauge transformation—the only difference being that for B fields, a "gauge transformation" is defined by a bundle.

Next, define

$$\omega_\alpha^{g,h} = \frac{h_\alpha^{g,h}}{\bar{h}_\alpha^{g,h}}. \tag{14}$$

From dividing the expressions

$$\nu_{\alpha\beta}^{gh} = (\nu_{\alpha\beta}^h)(h^* \nu_{\alpha\beta}^g)(h_\alpha^{g,h})(h_\beta^{g,h})^{-1}$$

$$\bar{\nu}_{\alpha\beta}^{gh} = (\bar{\nu}_{\alpha\beta}^h)(h^* \bar{\nu}_{\alpha\beta}^g)(\bar{h}_\alpha^{g,h})(\bar{h}_\beta^{g,h})^{-1}$$

we find that

$$T_{\alpha\beta}^{gh} = (T_{\alpha\beta}^h)(h^* T_{\alpha\beta}^g)(\omega_\alpha^{g,h})(\omega_\beta^{g,h})^{-1} \tag{15}$$

which means that the $\omega_\alpha^{g,h}$ are local-coordinate realizations [13], Sec. 5.5 of a map $\omega^{g,h}$ between bundles:

$$\omega^{g,h}: T^h \otimes h^* T^g \to T^{gh}. \tag{16}$$

We should take a moment to carefully explain what we mean by $\otimes$, since we have been describing the bundles T^g as principal bundles, not vector bundles. One way to understand $\otimes$ is to think of it as the product of Abelian torsors, following [14], Sec. 5.1. Alternatively, one could think of the T^g as complex line bundles with Hermitian fiber metrics. Perhaps the easiest way to understand $\otimes$ in the present context is

simply as the bundle whose transition functions are the product of the transition functions of the bundles appearing in the product.

Next, from

$$(h_\alpha^{g_1,g_2g_3})(h_\alpha^{g_2,g_3}) = (g_3^* h_\alpha^{g_1,g_2})(h_\alpha^{g_1g_2,g_3})$$

we derive the commutivity condition

$$
\begin{array}{ccc}
T^{g_3} \otimes g_3^*(T^{g_2} \otimes g_2^* T^{g_1}) & \xrightarrow{\omega^{g_1,g_2}} & T^{g_3} \otimes g_3^* T^{g_1 g_2} \\[6pt]
\omega^{g_2,g_3} \downarrow & & \downarrow \omega^{g_1 g_2,g_3} \\[6pt]
T^{g_2 g_3} \otimes (g_2 g_3)^* T^{g_1} & \xrightarrow{\omega^{g_1,g_2 g_3}} & T^{g_1 g_2 g_3}.
\end{array}
$$

To summarize our results so far, we have found that any two orbifold group actions on the same set of B fields differ by a set of principal $U(1)$ bundles T^g together with bundle morphisms $\omega^{g_1,g_2}\colon T^{g_2} \otimes g_2^* T^{g_1} \to T^{g_1 g_2}$. As mentioned before, this is to say that any two orbifold group actions on B fields differ by a gauge transformation, as a principal $U(1)$ bundle defines a gauge transformation of B fields.

Next, define

$$\Xi(g)^\alpha = \bar{\Lambda}(g)^\alpha - \Lambda(g)^\alpha. \tag{17}$$

From subtracting the expressions

$$g^* A^{\alpha\beta} = \begin{cases} A^{\alpha\beta} + d \log \nu_{\alpha\beta}^g + \Lambda(g)^\alpha - \Lambda(g)^\beta \\ A^{\alpha\beta} + d \log \bar{\nu}_{\alpha\beta}^g + \bar{\Lambda}(g)^\alpha - \bar{\Lambda}(g)^\beta \end{cases}$$

we find that

$$\Xi(g)^\alpha - \Xi(g)^\beta = d \log T_{\alpha\beta}^g. \tag{18}$$

In other words, the local one-forms $\Xi(g)^\alpha$ define a connection on the bundle T^g.

From subtracting

$$g^* B^\alpha = \begin{cases} B^\alpha + d\Lambda(g)^\alpha \\ B^\alpha + d\bar{\Lambda}(g)^\alpha \end{cases}$$

we see that

$$d\Xi(g)^\alpha = 0. \tag{19}$$

In other words, the connection $\Xi(g)^\alpha$ is not just any connection on the principal $U(1)$ bundle T^g, but must be a flat connection. The analogous statement in studying orbifold group actions on $U(1)$ gauge fields is that any two orbifold group actions differ by a constant gauge transformation.

From subtracting

$$\Lambda(g_1 g_2)^\alpha = \Lambda(g_2)^\alpha + g_2^* \Lambda(g_1)^\alpha - d \log h_\alpha^{g_1,g_2}$$

$$\bar{\Lambda}(g_1 g_2)^\alpha = \bar{\Lambda}(g_2)^\alpha + g_2^* \bar{\Lambda}(g_1)^\alpha - d \log \bar{h}_\alpha^{g_1,g_2}$$

we find that

$$\Xi(g_1 g_2)^\alpha = \Xi(g_2)^\alpha + g_2^* \Xi(g_1)^\alpha + d \log \omega_\alpha^{g_1,g_2} \tag{20}$$

which means that the bundle morphism $\omega^{g_1,g_2}\colon T^{g_2} \otimes g_2^* T^{g_1} \to T^{g_1 g_2}$ is constrained to preserve the connection on the bundles.

To summarize our progress so far, we have found that any two orbifold group actions on a set of B fields differ by a collection of principal $U(1)$ bundles T^g with flat connection $\Xi(g)$, together with connection-preserving bundle morphisms $\omega^{g_1,g_2}\colon T^{g_2} \otimes g_2^* T^{g_1} \to T^{g_1 g_2}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
T^{g_3} \otimes g_3^*(T^{g_2} \otimes g_2^* T^{g_1}) & \xrightarrow{\omega^{g_1,g_2}} & T^{g_3} \otimes g_3^* T^{g_1 g_2} \\[6pt]
\omega^{g_2,g_3} \downarrow & & \downarrow \omega^{g_1 g_2,g_3} \\[6pt]
T^{g_2 g_3} \otimes (g_2 g_3)^* T^{g_1} & \xrightarrow{\omega^{g_1,g_2 g_3}} & T^{g_1 g_2 g_3}.
\end{array} \tag{21}
$$

We should be careful at this point. Although we have not emphasized this point, it is only equivalence classes of bundles T^g with connection $\Xi(g)$ that are relevant. In a nutshell, if Λ and Λ' are two one-forms that differ by an exact form, then $B + d\Lambda = B + d\Lambda'$, so any two bundles with connection that differ by a gauge transformation (of the bundle) define the same action on the B field. So, if T'^g is another set of principal $U(1)$ bundles with connection $\Xi'(g)$, and $\kappa_g\colon T^g \to T'^g$ are connection-preserving bundle isomorphisms, then we can replace the data given above with the collection $(T'^g, \Xi'(g), \omega'^{g_1,g_2})$, where the ω' are given by

$$\omega'^{g_1,g_2} \equiv \kappa_{g_1 g_2} \circ \omega^{g_1,g_2} \circ (\kappa_{g_2} \otimes g_2^* \kappa_{g_1})^{-1} \tag{22}$$

to get an equivalent orbifold group action on the B fields.

C. $H^2(\Gamma, U(1))$

How do elements of $H^2(\Gamma, U(1))$ arise? Take the bundles T^g to be topologically trivial, and the connections $\Xi(g)$ to be gauge trivial. We can then map the bundles T^g to the canonical trivial bundle (whose transition functions are all identically 1), and gauge transform the connections $\Xi(g)$ to zero.

In this case, the bundle morphisms $\omega^{g_1,g_2}\colon T^{g_2} \otimes g_2^* T^{g_1}$ become gauge transformations of the canonical trivial bundle. From the fact that the ω^{g_1,g_2} must preserve the connection [i.e., Eq. (20)], and assuming the covering space is connected, we see that the gauge transformations ω^{g_1,g_2} must be constant gauge transformations.

We have reduced the data describing this set of B field gauge transformations to a set of maps $\omega\colon \Gamma \times \Gamma \to U(1)$. From commutivity of diagram (21), we see that the ω define a group 2-cocycle, i.e.,

$$\omega^{g_1,g_2 g_3} \omega^{g_2,g_3} = \omega^{g_1 g_2,g_3} \omega^{g_1,g_2}.$$

So far we have reduced the data describing this set of B field gauge transformations to a group 2-cocycle. More can be said: there is still a residual set of gauge transformations that must be taken into account. We can perform a constant

gauge transformation on each of the bundles T^g; this will preserve the connection on each bundle. From Eq. (22), we see that these constant gauge transformations simply change the group 2-cocycle ω^{g_1,g_2} by a coboundary.

Thus, we find that this set of B field gauge transformations is classified by elements of $H^2(\Gamma,U(1))$.

It should be emphasized that the appearance of $H^2(\Gamma,U(1))$ above holds regardless of whether or not Γ acts freely—nowhere have we made any assumptions concerning how the orbifold group Γ acts.

D. Detailed classification of orbifold group actions

In the previous section we showed how elements of $H^2(\Gamma,U(1))$ describe at least some differences between orbifold group actions. For $U(1)$ gauge fields, we found that all orbifold group actions differed by some element of $H^1(\Gamma,U(1))$—to what extent can the analogous statement be made here?

Do all orbifold group actions on B fields differ by such data described by $H^2(\Gamma,U(1))$? We shall argue that under special circumstances, all orbifold group actions on B fields differ by elements of $H^2(\Gamma,U(1))$, but in general there can be additional differences.

Suppose the covering space X is connected, simply connected, and $H^2(X,\mathbf{Z})$ has no torsion. Let $(T^g,\Xi(g),\omega^{g_1,g_2})$ be a set of data defining the difference between two orbifold group actions. We know that the connections $\Xi(g)$ on the bundles T^g are flat, which means that for each g, $c_1(T^g)$ must be a torsion element of $H^2(X,\mathbf{Z})$. However, we have assumed that $H^2(X,\mathbf{Z})$ has no torsion—so the bundles T^g must all be topologically trivial, i.e., $c_1(T^g)=0$ in $H^2(X,\mathbf{Z})$. In addition, we assumed the space X is simply connected. On a simply connected space, the only flat connections on a topologically trivial bundle are gauge trivial (gauge equivalent to the zero connection). So, if X is simply connected and $H^2(X,\mathbf{Z})$ has no torsion, then the bundles T^g are all topologically trivial and the connections $\Xi(g)$ are all gauge trivial. As noted in the last section, such gauge transformations of B fields are classified by $H^2(X,\mathbf{Z})$.

Thus, if the covering space X is simply connected and $H^2(X,\mathbf{Z})$ has no torsion, then any two orbifold group actions on a B field differ by a set of gauge transformations classified by an element of $H^2(\Gamma,U(1))$.

Suppose now that these criteria are not met—X is not simply connected, or $H^2(X,\mathbf{Z})$ has torsion. Then not all B-field gauge transformations need be described by topologically trivial bundles T^g with gauge trivial connections $\Xi(g)$—if X is not simply connected, then even on a topologically trivial bundle one can have flat connections which are not gauge trivial, and if $H^2(X,\mathbf{Z})$ has torsion, then one can have topologically nontrivial bundles with flat connections.

As a result, in general it appears that there can be additional orbifold group actions on B fields, beyond those classified by $H^2(\Gamma,U(1))$.

In retrospect, we should not have been surprised. Consider the special case in which Γ is freely acting and the B field is identically zero everywhere. We can compute the homology of the quotient X/Γ from the Cartan-Leray spectral sequence [16], Sec. VII.7:

$$E^2_{p,q}=H_p(\Gamma,H_q(X,\mathbf{Z}))\Rightarrow H_{p+q}(X/\Gamma,\mathbf{Z})$$

so (loosely ignoring differentials for simplicity) we see that $H_2(X/\Gamma,\mathbf{Z})$ ultimately receives contributions from not only $H_2(\Gamma,\mathbf{Z})$ [which dualizes to $H^2(\Gamma,U(1))$] and $H_2(X,\mathbf{Z})$, but also from $H_1(\Gamma,H_1(X,\mathbf{Z}))$—so if X is not simply connected, then one should expect additional contributions beyond those determined by $H^2(\Gamma,U(1))$ and $H_2(X,\mathbf{Z})$. (For a lengthier discussion of such cohomology calculations, and how $H^2(\Gamma,U(1))$ enters them, see [5].)

How should such additional orbifold group actions show up physically? In terms of twisted sector contributions to one-loop partition functions, for example, the orbifold group actions classified by $H^2(\Gamma,U(1))$ merely multiply twisted-sector contributions $Z_{(g,h)}$ by phases. We shall argue later that these additional orbifold group actions do more than just multiply $Z_{(g,h)}$ by a phase—these will deeply change $Z_{(g,h)}$ itself, by changing the weighting of individual sigma model map contributions to $Z_{(g,h)}$ in a winding-number-dependent fashion.

In fact, we shall argue in much greater detail elsewhere [15] that these "new" degrees of freedom are actually some very old degrees of freedom, the so-called "shift orbifolds" that play an important role in asymmetric orbifolds, but are rather more boring in symmetric orbifolds.

E. Commentary

So far we have described the set of orbifold group actions on B fields, described differences between any two orbifold group actions on a fixed B field, unveiled $H^2(\Gamma,U(1))$, and also discovered some new and more subtle orbifold group actions.

A few general comments are in order.

First, the results of this section do not depend upon Γ being freely acting. Everything we have described is the same for Γ having fixed points as for Γ freely acting—the details of the action of Γ on the base space are entirely irrelevant.

Second, the results of this section do not depend upon whether or not Γ is Abelian. We get the same results if Γ is non-Abelian.

Third, the results of this section do not depend upon whether or not the B field is flat or topologically trivial. In principle, precisely the same remarks hold regardless of whether $H=0$ or $H\neq0$ in $H^3(\mathbf{Z})$ on the covering space. However, if the B fields are described by a nonzero element of $H^3(\mathbf{Z})$—if the 1-gerbe is not topologically trivial—then one must check whether an orbifold group action actually exists, just as for orbifold $U(1)$ Wilson lines.

Now, we have described $H^2(\Gamma,U(1))$ as arising in the differences between two orbifold group actions, but that is not quite how people usually discuss it—people speak of "turning on" discrete torsion. This is because, just as for orbifold $U(1)$ Wilson lines, in almost every case in physics where orbifolds are studied, the B fields are such that there is
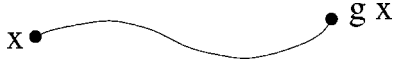
FIG. 1. Lift of closed loop to covering space.

a canonical trivial orbifold group action. For example, just as for orbifold $U(1)$ Wilson lines, if the B field is topologically trivial, then there is a notion of a canonical trivial action, and so we can define any orbifold group action in terms of its difference from the trivial action. Thus, in this case, we can indeed turn on discrete torsion.

In general, however, one cannot always expect to have such a canonical trivial lift. This matter is discussed further in [5].

IV. DERIVATION OF TWISTED SECTOR PHASES

In this section we shall derive the phases that appear in twisted sector contributions to partition functions, as originally described in [1].

These phases appear precisely because the string sigma model has a term

$$\int B.$$

On the covering space, a contribution to a twisted sector is a polygon with sides identified under the group action. The group action that identifies the sides lifts to an action on the B field; that action contributes a phase to

$$\exp\left(\int B \right)$$

and so the twisted sector contribution to the partition function comes with a phase.

We shall begin in Sec. IV A by discussing a simpler analogue of this behavior for orbifold Wilson lines. In Sec. IV B we shall derive Vafa's twisted sector phases for string one-loop partition functions, and in Sec. IV C we shall derive the twisted sector phases for string two-loop partition functions.

A. Analogue for orbifold $U(1)$ Wilson lines

To root ourselves, we shall begin by reviewing the analogue of twisted sector phases for orbifold Wilson lines. Consider computing a Wilson loop in some gauge theory [for simplicity, we shall assume a $U(1)$ gauge theory] on the quotient space. Suppose furthermore that such a loop descends from an open loop on the covering space, whose ends are identified by the action of some element g of the orbifold group Γ, as shown in Fig. 1. For simplicity we shall assume the bundle on which the connection lives is topologically trivial, and so has a canonical trivial orbifold group action—so we can specify any other orbifold group action in terms of a set of gauge transformations.

How do we calculate the value of the Wilson loop

$$\exp\left(\int A \right)$$

while working on the covering space?

In order to see what to do, first consider the problem of calculating the holonomy of a vector connection, in the case that A^α is only defined in patches along the loop. In such a case, one splits the loop into segments, each segment completely contained in a patch in which A^α is defined. Then, the holonomy is given by the product of $\exp(\int A)$ from each patch, separated by factors of the transition functions (evaluated at the borders of the segments). At the end of the day, one can check that this holonomy is independent of the precise splitting of the loop into segments.

Now, we shall return to the problem of calculating Wilson loops on covering spaces. First, there is a contribution to the Wilson loop from integrating the vector field A along the path from x to gx, i.e., there is a factor

$$\exp\left(\int_x^{gx} A \right). \tag{23}$$

However, this factor is not the end of the story. A Wilson loop (i.e., a Wilson line around a closed loop) on the quotient space will be invariant under gauge transformations, whereas the factor (23) does not appear invariant at all. To fix matters, use the relationship between the gauge field A at x and at gx, i.e.,

$$A_{gx} = g^* A_x = A_x + d \log \varphi^g$$

where φ^g is a $U(1)$-valued function defining a gauge transformation—rather, defining the action of the orbifold group.

To close the loop, we need to include the gauge transformation relating A_{gx} and A_x. The correct value of the Wilson loop, as calculated on the covering space, is

$$\varphi_x^g \exp\left(\int_x^{gx} A \right). \tag{24}$$

We shall follow a similar procedure in analyzing the phase factor $\exp(\int B)$ in twisted sectors. The naive integral of B over a polygon in the covering space is not sufficient; we must add gauge transformations along the boundary, which in this case are Wilson lines along the boundary. Furthermore, for B fields, those Wilson lines along the boundary are still not quite sufficient—we also need to account for bad behavior at the corners of the polygon.

B. One loop

The string orbifold one-loop partition function receives contributions not only from T^2's in the covering space, but also from configurations of strings that form T^2's on the quotient space, but only form open polygons on the covering space, as illustrated in Fig. 2.

Just as orbifold Wilson loops received an extra phase when edges are identified by group actions, so "orbifold Wilson surfaces"

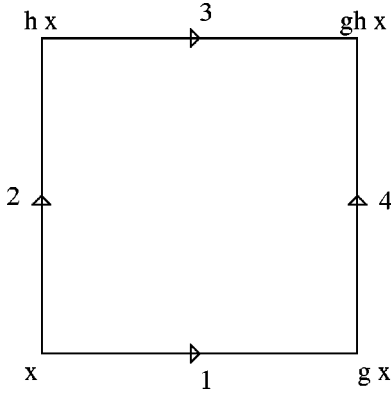$$\exp\left(\int B \right)$$

FIG. 2. A twisted sector contribution to the one-loop partition function.

get boundary contributions due to the orbifold action. In other words, naively integrating B over the region indicated will not give the phase over the cycle on the quotient space—we must also include contributions induced by gauge transformations of the B field occurring at the boundaries as the edges are glued together.

Before we begin the analysis, let us briefly pause to review how one calculates the holonomy of a B field over a surface, in the event that there is no two-form defined everywhere over the surface, but instead B is only defined on patches. To calculate the holonomy in this case, first tile the surface, in such a way that each tile is contained within a patch on which B^α is known, and each component of each tile boundary is contained within an overlap of patches. Then the holonomy of the B field is a product of several factors:

(1) For each tile, there is a factor of $\exp(\int B^\alpha)$, computed using the B field associated to the patch in which the tile lies.

(2) For each component of each tile boundary, there is a factor of $\exp(\int A^{\alpha\beta})$, where $B^\alpha - B^\beta = dA^{\alpha\beta}$.

(3) Finally, one can check that in order for the resulting holonomy to be independent of the choice of tiling, one must add factors of[4] $h_{\alpha\beta\gamma}$, evaluated at each point where multiple boundary components intersect.

At the end of the day, the result is independent of the choice of tiling. (Note that, when applied to Wess-Zumino-Witten (WZW) models, this gives us a means of understanding the exponential of the Wess-Zumino term that does not involve appealing to bounding three manifolds—after all, the exponential of the Wess-Zumino term is just the holonomy of the pullback of the B field on the group manifold. The fact that the Wess-Zumino term can change by integral amounts comes from possible gauge transformations of the B field.) After reflection, it is clear that a closely related procedure will allow us to calculate the orbifold Wilson surface holonomies, as we shall now demonstrate.

1. Basic analysis

To be specific, consider the region D shown in Fig. 2. Sides 1 and 3 are identified by $h \in \Gamma$, and sides 2 and 4 are

---

[4]Where $\delta A^{\alpha\beta} = d \log h_{\alpha\beta\gamma}$.

identified by $g \in \Gamma$. As a technical aside, note that, regardless of whether Γ is Abelian, one sums over contributions from commuting pairs (g,h) in describing twisted sectors, precisely so that the square shown in Fig. 2 will close at the upper right.

For simplicity we shall assume that[5] $B \equiv 0$. In this case, since there is a canonical trivial action of the orbifold group on the B fields, we can define any other action of the orbifold group on the B fields entirely in terms of the gauge transformations that distinguish it from the canonical trivial action. Let such a set of gauge transformations be denoted by bundles T^g with connection $\Lambda(g)$, and connecting bundle maps ω^{g_1, g_2}.

For the moment, we shall assume that the bundles T^g are the canonical trivial bundle (so the ω^{g_1, g_2} are all merely gauge transformations, not bundle morphisms), and the connections $\Lambda(g)$ are all gauge equivalent to the zero connection. We shall examine the more general case after examining this case in detail.

Now, under a gauge transformation by a bundle T with connection[6] Λ, the B field locally transforms as

$$B \mapsto B + d\Lambda$$

and so the holonomy of B over some surface S transforms as

$$\exp\left(\int_S B \right) \mapsto \exp\left(\int_S B \right) \exp\left(\int_{\partial S} \Lambda \right).$$

As a result, one naively would expect that the integral of B over region D would receive a contribution

$$\exp\left(\int_x^{hx} \Lambda(g) - \int_x^{gx} \Lambda(h) \right) \tag{25}$$

from the gauge transformations of the B field at the boundaries. (Relative signs are determined by a choice of orientation.)

However, Eq. (25) cannot be the correct answer. We mentioned earlier that we are free to gauge transform any of the bundles T^g—but expression (25) is not invariant under gauge transformations of the bundles.

In order to fix Eq. (25) so as to get a gauge-invariant result, we need to consider the gauge transformations at the corners. For example, the right end of side 3 in Fig. 2 is gauge transformed by $g * T^h$, whereas the top end of side 4 is gauge transformed by $h * T^g$, and as noted elsewhere, these need not be identical. Phrased differently, we need to find factors to add to expression (25) that soak up the gauge transformations of

---

[5]Also assume that the associated gerbe is topologically trivial.

[6]Previously we denoted the connection on bundle T^g by $\Xi(g)$, and used $\Lambda(g)$ to denote data used to define the action of the orbifold group on a B field. At this point we are changing notation—we will use $\Lambda(g)$ instead of $\Xi(g)$ to denote the connection on bundle T^g.

$$\Lambda(g)_{hx} - \Lambda(g)_x - \Lambda(h)_{gx} + \Lambda(h)_x$$
$$= h^*\Lambda(g)_x - \Lambda(g)_x - g^*\Lambda(h)_x + \Lambda(h)_x.$$

To fix these corner contributions, consider the following expression relating $\omega^{g,h}$ and the $\Lambda(g)$'s:

$$\Lambda(gh) = \Lambda(h) + h^*\Lambda(g) + d\log\omega^{g,h}.$$

By substracting this expression from the expression involving $\omega^{h,g}$, we find that

$$[\Lambda(h) + h^*\Lambda(g)] - [\Lambda(g) + g^*\Lambda(h)]$$
$$= d[\log\omega^{h,g} - \log\omega^{g,h}].$$

Using this it is clear that the corrected one-loop phase correction is given by

$$(\omega^{g,h}_x)(\omega^{h,g}_x)^{-1}\exp\left(\int_x^{hx}\Lambda(g) - \int_x^{gx}\Lambda(h)\right). \quad (26)$$

Our derivation leaves an overall constant factor ambiguous; this factor can be fixed by e.g., comparing to B-field holonomies on $T^n = \mathbf{R}^n/\mathbf{Z}^n$.

Under gauge transformations of the individual bundles T^g,

$$\Lambda(g) \mapsto \Lambda(g) + d\log\phi^g,$$

$$\log\omega^{g,h} \mapsto \log\omega^{g,h} + \log\phi^{gh} - \log\phi^h - h^*\log\phi^g,$$

and it is straightforward to check that Eq. (26) is indeed invariant under these gauge transformations.

How do the phases in Eq. (26) compare to the phases listed in [1]? It is straightforward to check that they are the same. Recall that in order to explicitly recover representative cocycles, one maps the topologically trivial bundles T^g to the canonically trivial bundle and gauge transformations the gauge-trivial connections $\Lambda(g)$ to the zero connection. The remaining ω^{g_1,g_2} are then constant gauge transformations. If we evaluate expression (26) in the described gauge, which we are free to do since expression (26) is gauge invariant, we see immediately that the phases associated to twisted sectors are given by

$$(\omega^{g,h})(\omega^{h,g})^{-1} \quad (27)$$

which are precisely the phases listed in [1] for a twisted sector contribution from a pair (g,h).

Since the phase in Eq. (27) depends only on g and h, and is independent of all other details of the polygon (such as the location of corners, winding numbers of the sides, and so forth), it is the same for all contributions to $Z_{(g,h)}$, and so multiplies $Z_{(g,h)}$ by an overall phase, precisely as observed in [1].

In passing, note that the phases (27) are invariant under changing the group cocycles by a group coboundary—indeed, they must be, otherwise we could not meaningfully associate phases to elements of $H^2(\Gamma, U(1))$.

To summarize our progress so far, we have just successfully derived the twisted sector phases appearing at one-loop in [1].

2. Invariant analysis

Next, we shall back up and redo this derivation somewhat more invariantly. In order to derive these phases, we took the bundles T^g to not only be topologically trivial, but actually the canonical trivial bundle, so that the bundle morphisms ω^{g_1,g_2} would be gauge transformations, which we used implicitly in writing Eq. (26). These phases can also be described more invariantly, which we shall now do.

To describe the phases more invariantly, note that the Wilson line

$$\int_x^{gx}\Lambda(h)$$

defines a map $T^h_x \to T^h_{gx} = g^*T^h_x$, so we can think of this Wilson line as defining an element of

$$(T^h_x)^\vee \otimes (g^*T^h_x)$$

and we can think of the difference of Wilson lines in Eq. (25) as an element of

$$[(T^g_x)^\vee \otimes (h^*T^g_x)] \otimes [(T^h_x)^\vee \otimes (g^*T^h_x)]^\vee \quad (28)$$

which is mapped by the composition of bundle morphisms $(\omega^{h,g})^{-1} \circ \omega^{g,h}$ to a scalar. So, if we interpret the naive difference of Wilson lines (25) as an element in Eq. (28), then we can write the T^2 (g,h) twisted sector phase as

$$(\omega^{h,g})^{-1} \circ (\omega^{g,h})\left[\int_x^{hx}\Lambda(g) - \int_x^{gx}\Lambda(h)\right]. \quad (29)$$

Note that expression (29) makes no assumptions concerning the nature of the bundles or the connections on them, unlike expression (26), and is well-defined under bundle isomorphisms.

3. Treatment of the shift orbifold degrees of freedom

In Sec. III, we argued that in addition to elements of $H^2(\Gamma, U(1))$, if the covering space X is not simply connected, or if there are torsion elements in $H^2(X, Z)$, then one can get additional possible gauge transformations of B fields, beyond those classified by elements of $H^2(\Gamma, U(1))$. Using the methods of this section, we can now see how such gauge transformations would appear in considering orbifold partition functions.

Suppose X is not simply connected, and one of the bundles T^g defining a gauge transformation of the B field is topologically trivial, but has a connection with nontrivial holonomy around some cycle. From Eq. (26), we see that in a (g,h) twisted sector, the phase will no longer be merely $\omega^{g,h}(\omega^{h,g})^{-1}$, but will also receive a winding-number-dependent contribution. In other words, instead of merely multiplying the (g,h) partition function $Z_{(g,h)}$ by a phase, the partition function $Z_{(g,h)}$ will itself be altered.
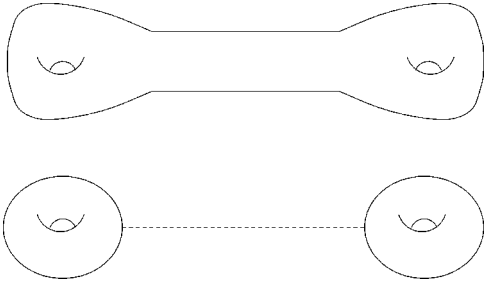
FIG. 3. Degeneration of a genus two Riemann surface.

In other words, a more detailed analysis shows that, in principle, there can be more degrees of freedom than those found in [1], which do more than just multiply twisted sector contributions by phases, but alter those twisted sector contributions.

We will argue in [15] that these degrees of freedom are not new, but rather are precisely shift orbifolds, which play an important role in asymmetric orbifolds, but are less interesting in symmetric orbifolds. In retrospect, one should have guessed that shift orbifolds would appear in our analysis—after all, conventional lore attributes shift orbifold degrees of freedom to the B field, in addition to discrete torsion.

C. Two loops

In this section we shall check our calculation of one-loop phase factors, by repeating the same calculation at the two-loop level. In particular, we shall explicitly verify factorization of the phase factors for two loops into a product of phase factors for one-loop diagrams.

We shall briefly review factorization, then we shall move on to the two-loop calculation.

1. Review of factorization

There is an old notion [17] that higher-loop string amplitudes are constrained by one-loop string amplitudes. This notion, known as factorization, was used in [1] to write down the phase factors for twisted sector contributions to higher-loop partition functions, in terms of the phase factors for one-loop partition functions.

For two-loop partition functions, the general idea can be expressed rather briefly. Since genus two Riemann surface can degenerate into a pair of genus one Riemann surfaces, separated by a long thin tube, as sketched in Fig. 3, and since the twisted sector phase is independent of the moduli of the Riemann surface, one finds that the twisted sector phase for a two-loop diagram must be a product of the twisted sector phases for each of the two one-loop diagrams appearing in the degeneration.

More specifically, if $\epsilon(g_1,h_1;g_2,h_2)$ denotes the twisted sector phase of a two-loop diagram determined by $g_1,h_1,g_2,h_2 \in \Gamma$, and this diagram can degenerate into a product of one-loop diagrams determined by (g_1,h_1), (g_2,h_2), with twisted sector phases $\epsilon(g_1,h_1)$ and $\epsilon(g_2,h_2)$, then [1]

$$\epsilon(g_1,h_1;g_2,h_2) = \epsilon(g_1,h_1)\epsilon(g_2,h_2). \tag{30}$$
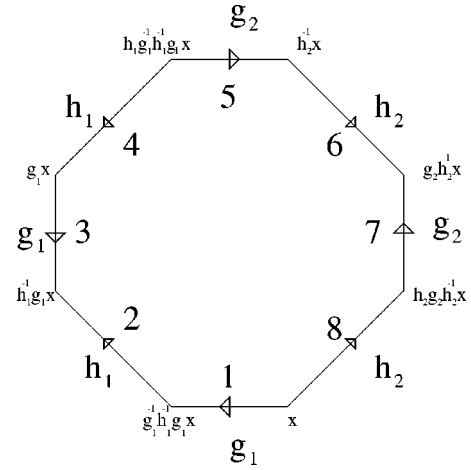

FIG. 4. A twisted sector contribution to the two-loop partition function.

In the following section we shall verify this factorization condition explicitly, by calculating the twisted sector phase $\epsilon(g_1,h_1;g_2,h_2)$ for a two-loop partition function, and checking that if this can degenerate into a product of two one-loop contributions, then Eq. (30) holds.

In order to perform the calculation in the following section, we shall assume $B \equiv 0$, as in the one-loop case, so that the action on the B field is completely determined by the gauge transformations (of B fields) distinguishing the action from the trivial action, and furthermore we shall restrict to gauge transformations determined by canonically trivial bundles T^g with gauge-trivial connections $\Lambda(g)$, i.e., those which correspond to elements of $H^2(\Gamma, U(1))$.

2. Two loop calculation

The string orbifold two-loop partition function receives contributions not only from genus two Riemann surfaces in the covering space, but also from configurations of strings that form genus two Riemann surfaces on the quotient space, but only form open polygons on the covering space, as illustrated in Fig. 4.

If two sides of the octagon in Fig. 4 are labeled with the same group element, it indicates that the two sides are identified under the action of that group element. For example, sides 1 and 3 are related by $g_1 \in \Gamma$, with orientations as indicated by the arrows.

In order for the sides of the octagon shown in Fig. 4 to close (i.e., in order to have an octagon), we must demand that

$$h_1 g_1^{-1} h_1^{-1} g_1 = g_2^{-1} h_2 g_2 h_2^{-1}. \tag{31}$$

In addition, we want to be able to deform this two-loop diagram into a pair of orbifold one-loop diagrams, determined by the pairs (g_1,h_1), (g_2,h_2), connected by a long thin tube. In order for those one-loop diagrams to be well defined (i.e., in order for the limit described to exist at all), we must demand that

$$g_1 h_1 = h_1 g_1 \quad \text{and} \quad g_2 h_2 = h_2 g_2.$$

Note that these constraints trivially satisfy Eq. (31). Also note that these constraints do not imply that every element of the set (g_1, h_1, g_2, h_2) need commute with every other element—we have not imposed any sort of commutivity condition relating g_1 and g_2, or h_1 and h_2, for example.

Following the same procedure as for the one-loop case, first note that the B fields on the sides are related by gluing as

$$B_3 = B_1 + d\Lambda(g_1)$$

$$B_4 = B_2 + d\Lambda(h_1)$$

$$B_7 = B_5 + d\Lambda(g_2)$$

$$B_8 = B_6 + d\Lambda(h_2).$$

From these identifications, we naively calculate that the phase defined by orbifold identifications is

$$\exp\left(\int_x^{g_1^{-1}h_1^{-1}g_1 x} \Lambda(g_1) + \int_{g_1^{-1}h_1^{-1}g_1 x}^{h_1^{-1}g_1 x} \Lambda(h_1) \right. $$
$$\left. + \int_{h_1 g_1^{-1}h_1 g_1 x}^{h_2^{-1}x} \Lambda(g_2) + \int_{h_2^{-1}x}^{g_2 h_2^{-1}x} \Lambda(h_2) \right). \quad (32)$$

As for the one-loop case, this cannot be the correct result, simply because it is not invariant under gauge transformations of the bundles $T^{g_1}, T^{h_1}, T^{g_2}, T^{h_2}$.

As for the one-loop case, we need to determine what to add to the expression above to correct the phases and fix the corner contributions. Specifically, the connections at the end points of the integrals in the naive phase calculation above are given by

$$(g_1^{-1}h_1^{-1}g_1)^*\Lambda(g_1)_x - \Lambda(g_1)_x + (h_1^{-1}g_1)^*\Lambda(h_1)_x$$
$$- (g_1^{-1}h_1^{-1}g_1)^*\Lambda(h_1)_x + (h_2^{-1})^*\Lambda(g_2)_x$$
$$- (g_2^{-1}h_2 g_2 h_2^{-1})^*\Lambda(g_2)_x + (g_2 h_2^{-1})^*\Lambda(h_2)_x$$
$$- (h_2^{-1})^*\Lambda(h_2)_x.$$

Define $y = g_1^{-1}h_1^{-1}g_1 x$ and $z = h_2^{-1}x$, then we can rewrite the connections at the integral endpoints as

$$\Lambda(g_1)_y - (g_1^{-1}h_1 g_1)^*\Lambda(g_1)_y + g_1^*\Lambda(h_1)_y - \Lambda(h_1)_y$$
$$+ \Lambda(g_2)_z - (g_2^{-1}h_2 g_2)^*\Lambda(g_2)_z + g_2^*\Lambda(h_2)_z - \Lambda(h_2)_z.$$

Next, using the relation

$$\Lambda(g) + g^*\Lambda(h) - \Lambda(h) - h^*\Lambda(g) = d(\log\omega^{g,h} - \log\omega^{h,g})$$

and the assumption that $g_1 h_1 = h_1 g_1$ and $g_2 h_2 = h_2 g_2$, we find that the naive expression can be corrected to give the gauge-invariant phase

$$(\omega_y^{g_1,h_1})(\omega_y^{h_1,g_1})^{-1}(\omega_z^{g_2,h_2})(\omega_z^{h_2,g_2})^{-1}$$

$$\times \exp\left(\int_x^{g_1^{-1}h_1^{-1}g_1 x} \Lambda(g_1) + \int_{g_1^{-1}h_1^{-1}g_1 x}^{h_1^{-1}g_1 x} \Lambda(h_1) \right.$$
$$\left. + \int_{h_1 g_1^{-1}h_1 g_1 x}^{h_2^{-1}x} \Lambda(g_2) + \int_{h_2^{-1}x}^{g_2 h_2^{-1}x} \Lambda(h_2) \right).$$

To evaluate the corrected expression above for the two-loop phase, gauge transform the connections $\Lambda(g)$ to be identically zero, so the gauge transformations $\omega^{g,h}$ become constant (assuming the covering space is connected, of course). Then we find that the correct expression for the two-loop phase is given by

$$(\omega^{g_1,h_1})(\omega^{h_1,g_1})^{-1}(\omega^{g_2,h_2})(\omega^{h_2,g_2})^{-1}. \quad (33)$$

This expression is precisely the product of the phases for the two one-loop contributions determined by (g_1, h_1) and (g_2, h_2). Thus, we have explicitly verified factorization at two loops, by explicitly calculating the twisted sector phase and noticing, at the end, that it factors, as desired.

Moreover, it is now patently obvious that by repeating the same calculation at any loop order, we will continue to find factorization—so our calculation of twisted sector phases obeys factorization at all loops.

V. DERIVATION OF ORBIFOLD GROUP ACTIONS ON D-BRANES

A description of discrete torsion for D-branes was proposed by Douglas and co-workers in [3,4]. Specifically, he proposed that turning on discrete torsion in a D-brane orbifold could be understood as replacing the honest representation of the orbifold group on the bundle on the D-brane, with a projective representation. This description is extremely natural—projective representations are classified by the same group cohomology group as discrete torsion, namely $H^2(\Gamma, U(1))$—and the author further justified his proposal by, for example, showing how the twisted sector phases of [1] could be derived from these projective representations. The association between discrete torsion and projective representations was further justified in [18], which gave additional evidence for the relationship.

In this section we shall give a first-principles derivation of these projective representations proposed in [3,4]. We shall also liberally use the conventions, notation, and general methods of Sec. III.

In a nutshell, because gauge transformations $B \mapsto B + d\Lambda$ are accompanied by $A \mapsto A + \Lambda$ on a D-brane worldvolume, in the presence of nontrivial gauge fields, the bundle on the worldvolume of the brane is twisted, and one is rapidly led to discover that orbifold group actions on such twisted bundles are (appropriately) projectivized. Such twisting has been pointed out explicitly in [19], and has also been used to describe K theory in B field backgrounds (Ref. [20], Sec. 5.3). See also [21] for another recent discussion of this twisting of D-brane gauge bundles by the B field. Twisted sheaves have appeared in [22] as part of an attempt to work out an

appropriate version of the generalized McKay correspondence [23] in the presence of discrete torsion.

In terms of the description of B fields involving stacks (see for example [6]), such twisted sheaves should be understood as special kinds of sheaves on stacks [24]. We shall not use such methods here, however.

For simplicity, we shall assume that the B field background is topologically trivial (i.e., that the curvature H is zero in integral cohomology). Our methods are very straightforward to generalize to $H \neq 0$, but the results are messier, and we see no need to include the generalization in this paper.

A. General analysis

Recall that in Sec. III we described B fields in terms of a collection of data $(B^\alpha, A^{\alpha\beta}, h_{\alpha\beta\gamma})$, where

$$B^\alpha - B^\beta = dA^{\alpha\beta}$$

$$A^{\alpha\beta} + A^{\beta\gamma} + A^{\gamma\alpha} = d\log h_{\alpha\beta\gamma}$$

$$\delta h_{\alpha\beta\gamma} = 1.$$

As in Sec. III, we shall work on a "good invariant" cover $\{U_\alpha\}$, meaning a cover that is well behaved with respect to the action of the orbifold group Γ.

Next, consider the gauge fields[7] A on a set of N coincident D-branes. Recall that in D-branes, gauge transformations of the B field and of the gauge fields are linked: $B \mapsto B + d\Lambda$ induces $A \mapsto A + I\Lambda$ [where I denotes the unit matrix, generating the overall $U(1)$ of $U(N)$].

In the present case, this means that since the B fields are only defined on local charts, the gauge fields A must also only be defined on local charts in general, with overlaps partially determined by the same gauge transformations relating the B fields on overlaps [19]. Specifically, to describe the gauge fields on a D-brane in the presence of B fields as above, we supplement the data for B fields by data for local $U(N)$ gauge fields A^α as

$$A^\alpha - g_{\alpha\beta} A^\beta g_{\alpha\beta}^{-1} - d\log g_{\alpha\beta}^{-1} = A^{\alpha\beta} I \qquad (34)$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = h_{\alpha\beta\gamma} I \qquad (35)$$

following [19]. As before, I is the unit matrix, generating the diagonal $U(1)$ in $U(N)$, $g_{\alpha\beta}$ is an invertible $N \times N$ matrix that would describe transition functions for the bundle if the B field were completely trivial, and A^α is a local $U(N)$ gauge field on the D-brane. Also note we are using $d\log g_{\alpha\beta}^{-1}$ as shorthand for $g_{\alpha\beta} dg_{\alpha\beta}^{-1}$.

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[7]Unfortunately, we shall use A to denote both the gauge field on the N D-branes, as well as part of the Čech–de Rham cocycle defining the B fields. These are distinct objects; our notation is unfortunate. The gauge fields will be associated with single elements of the cover $\{U_\alpha\}$, and so we shall usually denote the gauge fields by A^α. The Čech–de Rham cocycle elements live on intersections $U_\alpha \cap U_\beta = U_{\alpha\beta}$, and so we shall denote them by $A^{\alpha\beta}$. Hopefully the reader will not become too confused by this.

Now, ordinarily to describe how an orbifold group Γ acts on a bundle, we would demand

$$g^* g_{\alpha\beta} = (\gamma_\alpha^g)(g_{\alpha\beta})(\gamma_\beta^g)^{-1}$$

for some $N \times N$ matrices γ_α^g. However, because of Eq. (35), we have to be more careful. From studying the pullback of Eq. (35) by an element $g \in \Gamma$, we find

$$g^*[(g_{\alpha\beta})(g_{\beta\gamma})(g_{\gamma\alpha})] = (g^* h_{\alpha\beta\gamma})I = h_{\alpha\beta\gamma} \nu_{\alpha\beta}^g \nu_{\beta\gamma}^g \nu_{\gamma\alpha}^g I$$

where the ν^g are Čech cochains tha tappeared in Sec. II D in describing the orbifold group action on the B fields themselves. From the above, we see that the most general expression for $g^* g_{\alpha\beta}$ is given by

$$g^* g_{\alpha\beta} = [(\gamma_\alpha^g)(g_{\alpha\beta})(\gamma_\beta^g)^{-1}][(\nu_{\alpha\beta}^g)I] \qquad (36)$$

where the γ_α^g are some locally defined $U(N)$ adjoints. In short, we find that our naive expression for $g^* g_{\alpha\beta}$ is twisted.

From expressing $(g_1 g_2)^* g_{\alpha\beta}$ in two different ways (following the general self-consistency bootstrap outlined in Sec. III), we find a constraint on the γ^g:

$$(h_\alpha^{g_1,g_2})(\gamma_\alpha^{g_1 g_2}) = (g_2^* \gamma_\alpha^{g_1})(\gamma_\alpha^{g_2}). \qquad (37)$$

In fact, Eq. (37) above already tells us that ordinary lifts of orbifold group actions must be replaced by projective lifts, as hypothesized in [3,4], but we shall finish working out the orbifold group action on the D-brane gauge fields before emphasizing this point in detail.

Next, write

$$g^* A^\alpha = u(g)^\alpha A^\alpha [u(g)^\alpha]^{-1} + A(g)^\alpha$$

for some $U(N)$ adjoints $u(g)^\alpha$, $A(g)^\alpha$. This equation is not a constraint—it is general enough to describe any possible $g^* A^\alpha$, by varying $u(g)^\alpha$ and $A(g)^\alpha$. We are merely writing $g^* A^\alpha$ in a form that will yield more understandable results, following the self-consistent bootstrap of Sec. III. By pulling back both sides of Eq. (34) by $g \in \Gamma$ and examining the result, we can determine both $u(g)^\alpha$ and $A(g)^\alpha$:

$$u(g)^\alpha = \gamma_\alpha^g \qquad (38)$$

$$A(g)^\alpha = (\gamma_\alpha^g) d(\gamma_\alpha^g)^{-1} + I\Lambda(g)^\alpha \qquad (39)$$

where $\Lambda(g)^\alpha$ is a set of local one-forms that appeared in Sec. III in defining the action of the orbifold group on the B fields themselves.

To summarize, we have found that the orbifold group action on the $U(N)$ gauge fields on the worldvolume of a D-brane is described by

$$g^* A^\alpha = (\gamma_\alpha^g) A^\alpha (\gamma_\alpha^g)^{-1} + (\gamma_\alpha^g) d(\gamma_\alpha^g)^{-1} + I\Lambda(g)^\alpha$$

$$g^* g_{\alpha\beta} = (\nu_{\alpha\beta}^g)[(\gamma_\alpha^g)(g_{\alpha\beta})(\gamma_\beta^g)^{-1}]$$

$$(h_\alpha^{g_1,g_2})(\gamma_\alpha^{g_1 g_2}) = (g_2^* \gamma_\alpha^{g_1})(\gamma_\alpha^{g_2})$$

where $\Lambda(g)$ and $\nu^g_{\alpha\beta}$ were defined in Sec. III, in defining the action of the orbifold group on the B fields themselves.

As we shall work out explicitly below, the data above describes the projectivized orbifold group actions described in [3,4], but in considerably greater generality.

B. Explicit comparison to results of Douglas

The result above is considerably more general than that appearing in [3,4]. For purposes of comparison, let us take a moment to specialize to their circumstances. The papers [3,4] considered orbifolds of D-branes in backgrounds with vanishing B fields, where the D-branes had support on some flat space $\mathbf{R}^n$, and had a topologically trivial bundle on their worldvolume. Since the B fields are completely trivial, we can take $B^\alpha = 0$, $A^{\alpha\beta} = 0$, and $h_{\alpha\beta\gamma} = 1$.

Since the B fields are completely trivial, it is meaningful to speak of an honest bundle on the worldvolume of the D-brane—transition functions close on overlaps, not just up to a phase. Another assumption made in [3,4] is that this bundle on the D-brane worldvolume is topologically trivial. Since it is topologically trivial, we can take the $g_{\alpha\beta} = 1$, and replace the locally defined gauge fields A^α with a single global $U(N)$ gauge field A. In addition, [3,4] make the further assumption that the gauge field A is constant, so in defining the action of the orbifold group, it suffices to assume that γ^g is constant.

Since the B fields are trivial (in fact, vanishing), we can describe any orbifold group action on the B fields by specifying the difference between that action and the canonical trivial action. In other words, since the B field background is trivial, we can talk about orbifold group actions specified by elements of $H^2(\Gamma, U(1))$ (whereas, in general, only the difference between two orbifold group actions could be described in this fashion).

An orbifold group action specified by an element of $H^2(\Gamma, U(1))$ can be specified by a set of topologically trivial bundles T^g with gauge-trivial connection $\Lambda(g)$, and bundle maps ω^{g_1,g_2}. As noted earlier, without loss of generality we can take all the bundles T^g to be the canonical trivial bundle, and all the connections $\Lambda(g)$ to all be identically zero, so the ω^{g_1,g_2} become constant gauge transformations of the trivial bundle, satisfying the group 2-cocycle condition.

In terms of the data describing the orbifold group action on the D-brane worldvolume gauge fields, this means we can take the $\nu^g_{\alpha\beta} = 1$, and the h^{g_1,g_2}_α to be functions, i.e.,

$$h^{g_1,g_2}_\alpha = h^{g_1,g_2}_\beta \quad \text{on} \quad U_\alpha \cap U_\beta$$

and in fact constant functions, satisfying the group 2-cocycle condition

$$(h^{g_1,g_2g_3})(h^{g_2,g_3}) = (h^{g_1,g_2})(h^{g_1g_2,g_3}).$$

Since we assumed the B fields to be completely trivial, and we are describing all lifts in terms of gauge transformations combined with the canonical trivial lift (that exists in this case), the functions $h^{g,h}$ coincide with constant functions we

have labeled $\omega^{g,h}$ elsewhere—in other words, the 2-cocycles $h^{g,h}$ are the same cocycles as those describing the relevant element of $H^2(\Gamma, U(1))$.

We can now rewrite the action of the orbifold group on the D-brane gauge fields, in this special case, as

$$g * A = (\gamma^g) A (\gamma^g)^{-1}$$

$$(h^{g_1,g_2})(\gamma^{g_1g_2}) = (\gamma^{g_1})(\gamma^{g_2})$$

which is precisely the projectivized orbifold group action described in [3,4].

In recovering the form of the orbifold group action described in [3,4], we made the same simplifying assumptions that appeared in [3,4]—namely, that the D-branes reside on $\mathbf{R}^n$, and the bundle on the D-brane worldvolume is topologically trivial, with constant gauge field A. However, our results from the preceding section apply in much more general circumstances than these. Put another way, we have just shown how to derive the description of discrete torsion for D-branes given in [3,4], but our results apply in far greater generality than that used in [3,4].

C. Notes on shift orbifolds

In Sec. III we pointed out that there are additional orbifold group actions on B fields, in addition to those classified by elements of $H^2(\Gamma, U(1))$, which correspond to the so-called shift orbifolds [15]. As noted in Sec. IV, these new actions do considerably more than just multiply twisted sector partition functions $Z_{(g,h)}$ by a phase; they multiply individual contributions to $Z_{(g,h)}$ by distinct phases, and so they deeply alter $Z_{(g,h)}$.

How do these extra orbifold group actions appear on D-branes? The answer is implicit in the orbifold group action given at the end of Sec. V A above. Our results on D-brane actions made no assumptions regarding the form of the orbifold group action on the B fields. In the special case of orbifold group actions on trivial B fields arising from elements of $H^2(\Gamma, U(1))$, we re-derived the results of [3,4]; but our methods apply in general. We will describe the resulting group actions on D-branes in more detail in [15].

VI. NOTES ON VAFA AND WITTEN'S WORK

In [2] Vafa and Witten analyzed the inter-relationship between discrete torsion and Calabi-Yau moduli. Specifically, they considered deformations of the orbifolds $T^6/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ and $T^6/(\mathbf{Z}_3 \times \mathbf{Z}_3)$ both with and without discrete torsion. In both cases, turning on discrete torsion had the effect of removing most elements of $H^{1,1}_{orb}$ from the massless spectrum, while adding new elements to $H^{2,1}_{orb}$. In both cases they construct certain[8] families of Calabi-Yau deformations of the orbifolds, in which the elements of $H^{2,1}_{orb}$ form a subset of the possible polynomial deformations. Furthermore, they argue

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[8]For a more comprehensive description of Calabi-Yau deformations and resolutions of such orbifolds than was provided in [2], see [26], Sec. 6.

that the "allowed" complex structure deformations (those linked to elements of $H^{2,1}_{orb}$) cannot fully resolve the space, but always leave isolated singularities. Finally, they attempted to make sense out of these results by conjecturing the existence of some sort of "discrete torsion for conifolds," the isolated singularities left after desingularizing as much as possible.

How can one understand this behavior, in terms of our picture of discrete torsion? So far in this paper we have described discrete torsion as the action of the orbifold group on the B fields. In order to understand Ref. [2] we need to understand quotients of B fields, directly on singular spaces, and their behavior under deformation.

As always, there are many close analogues to properties of orbifold Wilson lines. We begin in Sec. VI A by describing the analogues of [2] in orbifold Wilson lines, where the analysis is vastly simpler. Then, in Sec. VI B we outline, in general terms, how the results of [2] can be understood in the framework we have presented, as (as always) an analogue for B fields of behavior of $U(1)$ gauge fields.

We shall not attempt to give a detailed first-principles derivation of the results in [2], but rather shall only outline general ideas. Detailed derivations are deferred to later work.

A. Analogue for orbifold Wilson lines

In this paper we have often discussed orbifold Wilson lines on smooth covering spaces. Understanding orbifold Wilson lines on singular quotient spaces is considerably more subtle.

To begin, consider ([25], Chap. 14) a trivial rank n complex vector bundle on $\mathbf{C}^2$. Let $\mathbf{Z}_2$ act on $\mathbf{C}^2$ in the standard fashion, and combine the $\mathbf{Z}_2$ action on the base with a gauge transformation that maps fibers to minus themselves, i.e.,

$$(z_1, z_2, \ldots, z_n) \mapsto (-z_1, -z_2, \ldots, -z_n).$$

Now, consider the quotient space. Over $\mathbf{C}^2/\mathbf{Z}_2$, we have some fibration, the quotient of the total space of the trivial $\mathbf{C}^n$ vector bundle. Specifically, over smooth points on $\mathbf{C}_2/\mathbf{Z}_2$, we recover the original $\mathbf{C}^n$ fiber, but over the singularity on $\mathbf{C}^2/\mathbf{Z}_2$, the fiber is $\mathbf{C}^n/\mathbf{Z}_2$.

In other words, the quotient of a bundle is not a bundle in general. We can understand this matter in somewhat less technical terms also. Consider quotienting $\mathbf{C}^2$ as above. Consider Wilson loops enclosing the origin in $\mathbf{C}^2/\mathbf{Z}_2$. At least for those loops descending from open strings on $\mathbf{C}^2$ with ends identified by the $\mathbf{Z}_2$, there will be a nontrivial holonomy about the loop. However, as we contract the loop to the origin, the holonomy remains nonzero—so something unusual must be going on at the origin. Since we started with a flat connection on the cover, the connection must be flat on the quotient space (at least, on the smooth part of the quotient). Since $\mathbf{C}^2/\mathbf{Z}_2$ is contractible, if there were an honest everywhere flat bundle on the space, any Wilson loop would be forced to be zero. Since there are nonzero Wilson loops, we are forced to conclude that our bundle must be behaving badly at the origin.

In order to understand orbifold Wilson lines on singular quotient spaces, one must work with more general objects than mere bundles. On complex surfaces, the relevant objects turn out to be reflexive, non-locally-free sheaves. On a smooth variety, a reflexive sheaf will be locally free up to complex codimension three, but on a singular variety, a reflexive sheaf can be locally free at lower codimension, provided the failure of local freedom occurs over the singularities. (As reflexive sheaves are not used very widely in the physics literature, we have included an appendix giving general background information on reflexive sheaves, as well as an example—a derivation of a reflexive rank 1 sheaf appearing as a $\mathbf{Z}_2$ quotient of $\mathbf{C}^2$ with nontrivial orbifold Wilson lines on the trivial line bundle over $\mathbf{C}^2$.)

If someone did not know about reflexive sheaves, and tried to understand the objects living on singular spaces, they would probably try to think of them as some analogue of orbifold Wilson lines. For example, there exist[9] reflexive non-locally-free sheaves at conifold singularities. Someone not acquainted with reflexive sheaves might try to label such objects as "orbifold Wilson lines for conifolds."

What happens to these reflexive sheaves when the space is resolved? On a complex surface, a reflexive sheaf will lift to a locally free sheaf, i.e., a bundle, on the resolved space. The new bundle will typically have nonzero curvature (nonzero c_1) associated with the exceptional divisor of the resolution. This can be seen directly in the ADHM/ALE construction, and incidentally forms one way of understanding[10] the classical McKay correspondence.

One can also understand this resolution in less technical terms. Consider our previous example of a quotiented bundle over $\mathbf{C}^2/\mathbf{Z}_2$. We remarked earlier that we have nonzero Wilson loops, when we naively would have expected all Wilson loops to be zero. Now, when we blow up the singularity, we can lift the Wilson loops to the cover. If we were to have an everywhere flat bundle on the resolution, then since $\pi_1 = 0$ and $H^2(\mathbf{Z})$ has no torsion, no Wilson loop can be nonzero—a contradiction. So, in order to avoid contradicting the existence of nonzero Wilson loops, we are forced to conclude that there must be nonzero curvature at some place, and as our connection was flat away from the singularity, the curvature must live on the exceptional divisor of the resolution. Phrased more naively, quotients of bundles will have nonzero curvature concentrated at curvature singularities if one turns on orbifold Wilson lines.

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[9] These exist mathematically, but whether they are relevant for physics is unknown.

[10] The classical McKay correspondence is a map from representations of Γ, a finite subgroup of $SU(2)$, to elements of the degree 2 cohomology of $\widetilde{\mathbf{C}^2/\Gamma}$. Here, we see that directly, a representation of Γ defines an action of the orbifold group on the trivial bundle on $\mathbf{C}^2$, and the corresponding element of degree 2 cohomology is c_1 of the bundle appearing as a lift of the reflexive sheaf on the quotient $\mathbf{C}^2/\Gamma$. This way of thinking about McKay is essentially due to [27,28].

B. *B* fields on singular spaces

We shall not attempt in this paper to give a thorough derivation of the results of [2], but in general terms, their results should now seem much more natural.

Just as a quotient of a bundle need not be a bundle, the quotient of a 1-gerbe (a formal structure, analogous to bundles, for which *B* fields are connections) need not be a 1-gerbe. Instead, on a singular variety, one would merely expect to have algebraic stacks (analogues of sheaves) which fail to be gerbes over certain quotient singularities.

For example, the discrete torsion for conifolds observed by the authors of [2] surely corresponds to some algebraic stack on a conifold singularity that fails to be a gerbe locally over the singularity, a precise analogue of the reflexive sheaves on conifolds that we labeled orbifold Wilson lines for conifolds.

We can justify this claim in elementary terms as follows. Recall in studying our example on $\mathbf{C}^2/\mathbf{Z}_2$, we concluded that because of the existence of nonzero Wilson loops on the quotient space, the resolved space must have nonzero curvature supported at the exceptional divisor. Here, we have a very similar situation. Here we have nonzero Wilson surfaces [i.e., $\exp(\int B)$] over Riemann surfaces in the quotient spaces $\mathbf{C}^3/(\mathbf{Z}_n \times \mathbf{Z}_n)$, as measured by the twisted sector phases worked out in Sec. III. Just as for orbifold Wilson lines, this is only sensible here because the spaces have singularities. If we resolve the singularities, in order to get a consistent picture we must generate nonzero curvature.

Now, in very general terms, there are two general ways to resolve singularities:

(1) We can deform the Kähler structure, by blowing up or making a small resolution.

(2) We can deform the complex structure.

In the first case, although we mentioned that we need nonzero curvature in order to make sense out of the Wilson surfaces present on the quotient, in general one does not generate any natural 3-cycle where the curvature H could be supported, since blowups and small resolutions add even-dimensional cycles. Thus, we are naturally led to the conclusion that, in general terms, resolutions of this class must be obstructed—we need to turn on curvature H somewhere to account for the nonzero Wilson surfaces, but typically one does not have any natural options to do so. Indeed, in [2] it was noted that nontrivial Kähler deformations are removed from the massless spectrum.

In the second case, one can often get 3-cycles. For example, if we smooth a conifold singularity by deforming the complex structure, then we are typically led to a new 3-cycle. So, in this case one expects to typically be able to resolve the space without contradiction, and generate nonzero curvature H on new 3-cycles.

Now, nonzero curvature H (in de Rham cohomology) on a Calabi-Yau is inconsistent with supersymmetry—if a deformation involves turning on H, then that deformation will break supersymmetry, and so that deformation is lifted from the moduli space of supersymmetric vacua. Indeed, in [2] it was found that complex structure deformations which completely resolved the space, and left no singularity, were also obstructed.

This explanation of how Calabi-Yau moduli of type $H^{2,1}$ could be obstructed was suggested in [2], Sec. 2.3 as an explanation of their results, though they had no idea why H should be generated. In the present context, this is naturally understood.

So, we see that if we try to deform an orbifold with non-zero discrete torsion (which implies nonzero Wilson surfaces on the quotient), then logical consistency and the demands of supersymmetry place strong constraints on the structure of the moduli space of vacua, which in general terms is consistent with [2].

As mentioned earlier, we shall not attempt a thorough derivation of the results of [2] in this paper, but rather shall defer such detailed considerations to future work. For the purposes of this paper, we are content to merely outline how the results of [2] naturally fit into the general picture we have described.

VII. NOTES ON LOCAL ORBIFOLDS

A point usually glossed over in physics discussions of orbifolds is the possibility of local orbifold degrees of freedom, as distinct from global orbifold degrees of freedom. We shall take a moment to very briefly discuss local orbifolds and discrete torsion, before concluding.

What are local orbifold degrees of freedom? Consider forming a global orbifold, say, $T^4/\mathbf{Z}_2$. In forming the global orbifold, if we are doing physics, then we usually get to choose some number of degrees of freedom—orbifold Wilson lines and discrete torsion, for example. Now, we can also examine each singularity locally. In the present case, each singularity is locally $\mathbf{C}^2/\mathbf{Z}_2$. If we forget about the global structure of the orbifold and just work locally, then one is led to associate orbifold degrees of freedom to the singularities locally. We refer to such local degrees of freedom as local orbifold degrees of freedom.

How do local orbifold degrees of freedom differ from global orbifold degrees of freedom? For simplicity, consider a $U(1)$ bundle with connection on T^4, and quotient by $\mathbf{Z}_2$. If we turn on the nontrivial orbifold $U(1)$ Wilson line, then we find that each $\mathbf{C}^2/\mathbf{Z}_2$ has a nontrivial "twist" (technically, locally we have a reflexive nonlocally-free rank 1 sheaf). So, the global orbifold gives rise to either a twist at every singularity, or no twists at any singularity.

However, there are additional options present if we consider the orbifold to be a local orbifold. We can consistently twist at groups of eight of the sixteen singularities (technically, in addition to reflexive non-locally-free sheaves that fail to be locally free at every singularity or nowhere, there are also reflexive non-locally-free sheaves that fail to be locally free at some but not all singularities). (See [29], Sec. VIII.5 or [30] for the mathematical result.)

One can now ask: does discrete torsion have local orbifold degrees of freedom? Previously, when discrete torsion was known merely as some mysterious discrete degree of freedom appearing in orbifolds, the answer was not known.

Now that we have a purely mathematical understanding of discrete torsion, we can address this matter. In general terms, it is now clear that there should exist local orbifold degrees

of freedom for discrete torsion. We shall not attempt any counting of such local orbifold degrees of freedom here—rather, our purpose was merely to emphasize the important but usually neglected point that local orbifold degrees of freedom exist.

VIII. CONCLUSIONS

In this paper we have given a complete geometric explanation of discrete torsion, as the choice of orbifold group action on the B fields. Specifically, we have shown how the group cohomology group $H^2(\Gamma, U(1))$ arises, derived the phases associated with twisted sector contributions to string loop partition functions, derived Douglas's description [3,4] of orbifold group actions on D-branes as projective representations of the orbifold group, and outlined how the results of Vafa and Witten [2] fit into this general framework. We have also briefly discussed shift orbifolds, which are degrees of freedom associated with the B field, beyond those classified by $H^2(\Gamma, U(1))$, and explained how these appear physically in terms of twisted sector contributions to partition functions and in terms of D-brane actions.

Nowhere in all this did we assume that the orbifold group Γ acts freely; nor do we assume that Γ is Abelian. We do not even assume that the curvature of the B fields vanishes. Our results hold in generality.

To put a different spin on these matters, we have given a completely geometric description of discrete torsion. *A priori*, discrete torsion has nothing to do with string theory. Discrete torsion is a property of defining orbifold group actions on B fields, and can be understood in a purely mathematical context, without any reference to string theory. Now, at the end of the day, we can calculate twisted sector contributions to partition functions, as well as check Douglas's proposed orbifold group action on D-branes, so we can certainly derive physical results. However, it should be emphasized that discrete torsion is not some special, "inherently stringy" property of string theory or conformal field theory, but rather has a straightforward and purely mathematical understanding.

ACKNOWLEDGMENTS

APPENDIX: REFLEXIVE SHEAVES ON QUOTIENT SPACES

In Sec. VI A, we mentioned that when orbifolding a space with a bundle, the resulting object living on the quotient space need not be a bundle, and in the context of complex algebraic geometry, will be a reflexive sheaf. In this appendix we elaborate on these remarks, as reflexive sheaves are not commonly understood in the physics literature. We begin with a short overview of reflexive sheaves, and then explicitly derive the reflexive sheaf appearing in the specific example of the quotient $\mathbf{C}^2/\mathbf{Z}_2$.

1. Technical notes on reflexive sheaves

In this appendix we shall give some technical notes on reflexive sheaves.

A reflexive sheaf is a sheaf $\mathcal{E}$ which is isomorphic to its bidual: $\mathcal{E} \cong \mathcal{E}^{\vee\vee}$. Reflexive sheaves include locally free sheaves (i.e., bundles) as a special subclass.

On smooth varieties, reflexive sheaves are locally free up to (complex) codimension three. Thus, for example, on a smooth surface, all reflexive sheaves are locally free. Also, on a smooth variety, all reflexive rank 1 sheaves are locally free.

On singular varieties, both of the statements above fail. On a singular surface, one can have reflexive non-locally-free sheaves; the sheaves fail to be locally free over the singularities. One can also have reflexive non-locally-free rank 1 sheaves on singular varieties.

On Noetherian normal varieties (i.e., most of the varieties that physicists are likely to encounter in practice), we can describe reflexive rank 1 sheaves in terms of divisors, just as line bundles on smooth varieties can be described in terms of divisors. Roughly speaking, a general divisor on a Noetherian normal variety is known as a "Weil" divisor, whereas a divisor that defines, not just a reflexive rank 1 sheaf, but a line bundle, is known as a "Cartier" divisor.[11]

For example, on the singular affine space $\mathbf{C}^2/\mathbf{Z}_2$, there is precisely one (equivalence class of) Weil divisor that is not Cartier. If we write

$$\mathbf{C}^2/\mathbf{Z}_2 = \text{Spec } \mathbf{C}[x,y,z]/(xy - z^2)$$

then the non-Cartier Weil divisor is $\{x = z = 0\}$. We shall denote this divisor by D. Then, the associated reflexive sheaf $\mathcal{O}(D)$ [and also $\mathcal{O}(-D)$] is not locally free.

Although D is not Cartier, it can be shown that the divisor $2D$ is Cartier, so for example, the reflexive sheaves $\mathcal{O}(-D)$ and $\mathcal{O}(D)$ are related by tensoring with a (trivial) line bundle, and so are in the same equivalence class of Weil divisors.

Similarly, the divisor $\{y = z = 0\}$ is not Cartier, but it is related to D by a Cartier divisor, and so lies in the same equivalence class of Weil divisors.

Reflexive rank 1 sheaves can be somewhat more subtle than line bundles. For example, if D_1 and D_2 are a pair of Cartier divisors, then as everyone knows,

$$\mathcal{O}(D_1) \otimes \mathcal{O}(D_2) = \mathcal{O}(D_1 + D_2).$$

In other words, the tensor products of the line bundles associated to (Cartier) divisors D_1, D_2 is the line bundle associated to (Cartier) divisor $D_1 + D_2$. However, if D_1 and D_2 are not both Cartier divisors, then this relation need not hold ([31], p. 283). In other words, if D_1 and D_2 are two Weil

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[11]Experts will note we are being quite sloppy—our description of Cartier divisors really corresponds to the image of Cartier divisors in the space of Weil divisors. Our abbreviated description will suffice for the purposes of this appendix.

divisors, then in general the reflexive rank 1 sheaves $\mathcal{O}(D_1)$, $\mathcal{O}(D_2)$ are not related to $\mathcal{O}(D_1+D_2)$ as above:

$$\mathcal{O}(D_1)\otimes\mathcal{O}(D_2)\neq\mathcal{O}(D_1+D_2).$$

The essential difficulty is that the tensor product of two reflexive sheaves need not be reflexive—the tensor product can contain torsion, for example. We can fix this problem by bidualizing the left-hand side of the expression above. In other words, a statement that is true for all Weil divisors D_1, D_2, not just Cartier divisors, is

$$[\mathcal{O}(D_1)\otimes\mathcal{O}(D_2)]^{\vee\vee}=\mathcal{O}(D_1+D_2).$$

A more detailed discussion of the relationship of Weil versus Cartier divisors, in the context of toric varieties, can be found in [10].

2. Reflexive sheaves and orbifold Wilson lines

In the text we claimed that one can directly check that possible quotients of the structure (trivial rank one) sheaf on $\mathbf{C}^2$ are precisely the possible reflexive sheaves on $\mathbf{C}^2/\Gamma$. Here we shall work this out in detail for the case $\Gamma=\mathbf{Z}_2$.

Let R denote the ring $\mathbf{C}[x,y]$, and let R^Γ denote the ring of Γ invariants, namely

$$R^\Gamma=\mathbf{C}[x^2,y^2,xy].$$

Let M denote the R module defining the structure sheaf. In other words, $M=R$. (The distinction between the module M and the ring R shall be important when discussing nontrivial lifts of Γ.)

First, consider the case of the trivial lift of the $\Gamma=\mathbf{Z}_2$ action from $\mathbf{C}^2$ to the structure sheaf. Then the module $M^{\tilde{\Gamma}}$ of Γ invariants is isomorphic (as an R^Γ module) to the trivial R^Γ module, namely

$$M^\Gamma=\mathbf{C}[x^2,y^2,xy].$$

In this case, M^Γ defines the structure sheaf on the quotient space $\mathbf{C}^2/\Gamma$. This sheaf is locally free.

Now, consider the nontrivial lift of the $\Gamma=\mathbf{Z}_2$ action from $\mathbf{C}^2$ to the structure sheaf on $\mathbf{C}^2$. In order to describe the action of this lift on the module M, view M as a freely generated R module, and let the (single) generator of M be denoted by α. For the trivial lift, we implicitly assumed that Γ mapped $\alpha\mapsto\alpha$. For the nontrivial lift, we take $\alpha\mapsto-\alpha$. The R^Γ module of Γ invariants of M is now a module with two generators, namely $x\alpha$ and $y\alpha$, and one relation

$$(xy)\cdot(x\alpha)=(x^2)\cdot(y\alpha).$$

The module M^Γ is not freely generated, and in particular does not define a locally free sheaf on $\mathrm{Spec}R^\Gamma=\mathbf{C}^2/\Gamma$. However, the module M^Γ does define a reflexive rank 1 sheaf.

Thus, we see explicitly that the nontrivial orbifold Wilson line on $\mathbf{C}^2/\mathbf{Z}_2$ describes a reflexive non-locally-free rank 1 sheaf on $\mathbf{C}^2/\mathbf{Z}_2$.

The reflexive, non-locally-free sheaf on $\mathbf{C}^2/\mathbf{Z}_2$ can be (non-uniquely) lifted to a sheaf $\mathcal{S}$ on the resolution of $\mathbf{C}^2/\mathbf{Z}_2$, such that (essentially) $(\pi_*\mathcal{S})^{\vee\vee}$ is the nontrivial reflexive sheaf on $\mathcal{C}^2/\mathbf{Z}_2$.

For those readers acquainted with toric varieties, the discussion above can be understood torically. Describe $\mathbf{C}^2/\mathbf{Z}_2$ by a cone with edges

$$v_1=(2,1)$$

$$v_2=(0,1)$$

and let D_1,D_2 denote the toric divisors corresponding to edges v_1,v_2, respectively. One can then check explicitly (using, for example, methods discussed in [10]) that the modules describing the reflexive rank 1 sheaves $\mathcal{O}(\pm D_1)$ and $\mathcal{O}(\pm D_2)$ are all isomorphic (as modules, ignoring the T grading), and in particular are isomorphic to the module M^Γ of invariants from the nontrivial $\mathbf{Z}_2$ lift, as described above. [Note that since there is only one reflexive non-locally-free sheaf, the sheaves $\mathcal{O}(+D_i)$ and $\mathcal{O}(-D_i)$ are necessarily isomorphic.]

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